

WAVE INVARIANTS AT ELLIPTIC CLOSED GEODESICS

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0. INTRODUCTION

The purpose of this article is to provide an effective method for calculating the wave invariants associated to a non-degenerate elliptic closed geodesic γ of a compact Riemannian manifold (M, g) : that is, the coefficients in the singularity expansion

$$\text{Tr}U(t) = e_o(t) + \sum_{\gamma} e_{\gamma}(t)$$

$$e_{\gamma}(t) \sim a_{\gamma-1}(t - L_{\gamma} + i0)^{-1} + \sum_{k=0}^{\infty} a_{\gamma k}(t - L_{\gamma} + i0)^k \log(t - L_{\gamma} + i0)$$

of the trace of the wave group $U(t) := \exp(it\sqrt{\Delta})$ at $t = L_{\gamma}$ (the length of γ .) We will show that

$$a_{\gamma k} = \int_{\gamma} I_{\gamma k}(s) ds$$

for certain homogeneous invariant densities $I_{\gamma k}(s)ds$ on γ , given by at most $2k+1$ integrals over γ of polynomials in the curvature, Jacobi fields, length, inverse length and Floquet invariants $\beta_j := (1 - e^{i\alpha_j})^{-1}$ along γ . These expressions characterize the wave invariants in much the same way that the heat invariants (or wave invariants at $t=0$) are characterized as integrals $\int_M P_j(R, \nabla R, \dots) d\text{vol}$ of homogeneous curvature polynomials over M [ABP] [Gi]. Moreover, in combination with the recent inverse results of Guillemin [G.1,2], the method produces a list of new spectral invariants of this kind, simpler than the wave invariants themselves (the so-called quantum Birkhoff normal form coefficients $B_{\gamma k; j}$.)

To state the results, we will need some notation. We let $\mathcal{J}_{\gamma}^{\perp} \otimes \mathbb{C}$ denote the space of complex normal Jacobi fields along γ , a symplectic vector space of (complex) dimension $2n$ ($n = \dim M - 1$) with respect to the Wronskian

$$\omega(X, Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincare map P_{γ} is then the linear symplectic map on $\mathcal{J}_{\gamma}^{\perp} \otimes \mathbb{C}$ defined by $P_{\gamma}Y(t) = Y(t + L_{\gamma})$. We will assume γ to be non-degenerate elliptic, i.e. that the eigenvalues of P_{γ} are of the form $\{e^{\pm i\alpha_j}, j = 1, \dots, n\}$ with (Floquet) exponents $\{\alpha_1, \dots, \alpha_n\}$, together with π , independent over \mathbb{Q} . The associated normalized eigenvectors will be denoted $\{Y_j, \overline{Y}_j, j = 1, \dots, n\}$,

$$P_{\gamma}Y_j = e^{i\alpha_j}Y_j \quad P_{\gamma}\overline{Y}_j = e^{-i\alpha_j}\overline{Y}_j \quad \omega(Y_j, \overline{Y}_k) = \delta_{jk}$$

and relative to a fixed parallel normal frame $e(s) := (e_1(s), \dots, e_n(s))$ along γ they will be written in the form $Y_j(s) = \sum_{k=1}^n y_{jk}(s)e_k(s)$. The metric coefficients g_{ij} will always be taken relative to Fermi normal coordinates (s, y) along γ . The m th jet of g along γ will be denoted by $j_{\gamma}^m g$, the curvature tensor by R and its covariant derivatives by $\nabla^m R$. The vector fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial y_j}$ and their real linear combinations will be referred to as Fermi normal vector fields along γ and contractions of tensor products of the $\nabla^m R$'s with these vector fields will be referred to as *Fermi curvature polynomials*. Such polynomials will be called *invariant* if they are invariant under the action of $O(n)$ in the normal spaces. Invariant contractions against $\frac{\partial}{\partial s}$ and against the Jacobi eigenfields Y_j, \overline{Y}_j , with coefficients given by invariant polynomials in the components y_{jk} , will be called *Fermi-Jacobi polynomials*. We will also use this term for functions on γ given by repeated

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indefinite integrals over γ of such FJ polynomials. Finally, FJ polynomials whose coefficients are given by polynomials in the Floquet invariants $\beta_j = (1 - e^{i\alpha_j})^{-1}$ will be called *Fermi-Jacobi-Floquet* polynomials. We give weights to the variables $g_{ij}, D_{s,y}^\beta g_{ij}, L := L_\gamma, \alpha_j, y_{ij}, \dot{y}_{ij}$ as follows: $wgt(D_{s,y}^\beta g_{ij}) = -|\beta|, wgt(L) = 1, wgt(\alpha_j) = 0, wgt(y_{ij}) = \frac{1}{2}, wgt(\dot{y}_{ij}) = -\frac{1}{2}$. As will be seen, these weights reflect the scaling of these objects under $g \rightarrow \epsilon^2 g$. A polynomial in this data is homogeneous of weight s if all its monomials have weight s under this scaling.

Theorem A *Let γ be an elliptic closed geodesic with $\{\alpha_1, \dots, \alpha_n, \pi\}$ independent over \mathbf{Q} . Then $a_{\gamma k} = \int_\gamma I_{\gamma k}(s; g) ds$ where :*

- (i) $I_{\gamma k}(s, g)$ is a homogeneous Fermi-Jacobi-Floquet polynomial of weight $-k - 1$ in the data $\{L, y_{ij}, \dot{y}_{ij}, D_{s,y}^\beta g\}$ with $|\beta| \leq 2k + 4$;
- (ii) The degree of $I_{\gamma k}$ in the Jacobi field components is at most $6k + 6$;
- (iii) At most $2k + 1$ indefinite integrations over γ occur in $I_{\gamma k}$;
- (iv) The degree of $I_{\gamma k}$ in the Floquet invariants β_j is at most $k + 2$.

For instance, in dimension 2 the residual wave invariant $a_{\gamma o}$ is given by:

$$a_{\gamma o} = \frac{c_\gamma}{L^\#} [B_{\gamma o; 4}(2\beta^2 - \beta - \frac{3}{4}) + B_{\gamma o; 0}]$$

where:

- (a) c_γ is the principal wave invariant $i^\sigma L^\# |I - P_\gamma|^{-\frac{1}{2}}$;
- (b) $L^\#$ the primitive length of γ ; σ is its Morse index; P_γ is its Poincare map;
- (c) $B_{\gamma o; j}$ have the form:

$$\begin{aligned} B_{\gamma o; j} = & \frac{1}{L^\#} \int_o^{L^\#} [a |\dot{Y}|^4 + b_1 \tau |\dot{Y} \cdot Y|^2 + b_2 \tau \text{Re}(\bar{Y} \dot{Y})^2 + c \tau^2 |Y|^4 + d \tau_{\nu\nu} |Y|^4 + e \delta_{jo} \tau] ds \\ & + \frac{1}{L^\#} \sum_{0 \leq m, n \leq 3; m+n=3} C_{1; mn} \frac{\sin((n-m)\alpha)}{|(1 - e^{i(m-n)\alpha})|^2} \left| \int_o^{L^\#} \tau_\nu(s) \bar{Y}^m \cdot Y^n(s) ds \right|^2 \\ & + \frac{1}{L^\#} \sum_{0 \leq m, n \leq 3; m+n=3} C_{2; mn} \text{Im} \left\{ \int_o^{L^\#} \tau_\nu(s) \bar{Y}^m \cdot Y^n(s) \left[\int_o^s \tau_\nu(t) \bar{Y}^n \cdot Y^m(t) dt \right] ds \right\} \end{aligned}$$

for various universal (computable) coefficients. Here,

- (d) τ denotes the scalar curvature, τ_ν its unit normal derivative, $\tau_{\nu\nu}$ the Hessian $\text{Hess}(\tau)(\nu, \nu)$; Y denotes the unique normalized Jacobi eigenfield, \dot{Y} its time-derivative and δ_{jo} the Kronecker symbol (1 if $j=0$ and otherwise 0.)

We note that the residual wave invariant already saturates the description in Theorem A.

This characterization of the wave invariants makes more concrete (for Laplacians) the recent results of Guillemin [G1,2] which show that the wave invariants may be expressed in terms of the quantum Birkhoff normal form coefficients of the wave operator around γ . For instance, Guillemin shows [G.2, (8.24)]:

$$\begin{aligned} a_{\gamma o} = & i c_\gamma \sigma \left[\sum_{i \neq j} \frac{\partial^2}{\partial I_i \partial I_j} H_1(0, \sigma) (\beta_i + \frac{1}{2}) (\beta_j + \frac{1}{2}) \right. \\ & \left. + \frac{1}{2} \sum_i (2\beta_i^2 + \beta_i - \frac{1}{4}) \frac{\partial^2}{\partial I_i^2} H_1(0, \sigma) - \sum_i (\beta_i + \frac{1}{2}) \frac{\partial}{\partial I_i} H_o(0, \sigma) + H_{-1}(0, \sigma) \right] \end{aligned}$$

where $H_{1-r}(\sigma, I_1, \dots, I_n)$ is the term of order $1-r$ in the complete symbol of the quantum normal form. The coefficients $B_{\gamma o; j}$ above are essentially these QBNF (quantum Birkhoff normal form) coefficients.

Indeed, the first step in the proof of Theorem A (which we call Theorem B) consists in explicitly constructing the QBNF for $\sqrt{\Delta}$ near γ . The method is different from that in [G.1,2] and effectively calculates the QBNF coefficients as integrals of Fermi-Jacobi polynomials. It is based to some extent on the construction of a complete set of quasi-modes associated to γ as presented in Babich-Buldyrev [B.B] although the emphasis is on intertwining operators rather than on quasi-modes per se. In the construction of the intertwining

operators it also employs several ideas in Sjostrand [Sj], although it does not begin by putting the principal symbol in Birkhoff normal form as is done in [Sj] and [G.2].

The second step (which we call Theorem C) consists in calculating the wave invariants from the normal form. This is possible because the wave invariants are non-commutative residues of the wave operator and its t derivatives and hence are invariant under conjugation by unitary Fourier Integral operators (cf. [G.1,2], [Z.1]). The residues of the normal form wave group will be easily seen to be polynomials in the QBNF coefficients and in the β_j 's. Some dimensional analysis of the wave coefficients then leads to the description in the statement of Theorem A.

Guillemin [loc.cit] has also proved the remarkable inverse result that, conversely, the QBNF coefficients can be determined from the wave invariants associated to γ and its iterates, and therefore are themselves spectral invariants. In view of Theorems A-B this gives a list of new spectral invariants, for instance the QBNF coefficients $B_{\gamma o; j}$ above, in the form of geodesic integrals of FJ polynomials,

Let us now describe the ingredients of the proofs more precisely. Henceforth we will reserve the notation γ for a primitive closed geodesic and will denote its iterates by γ^m . To introduce the quantum Birkhoff normal form at γ and its role in the calculations, we note first that the wave invariants associated to γ are determined by the microlocalization of Δ to a conic neighborhood

$$(0.1) \quad |y| < \epsilon \quad \frac{|\eta|}{\sigma} < \epsilon$$

of the cone $\mathbb{R}^+ \gamma$ thru γ in $T^*M - 0$. Here, (σ, η) denote the symplectically dual coordinates to the Fermi normal coordinates (s, y) above. The microlocalization of Δ to (0.1) is then given by

$$\Delta_\psi := \psi(s, D_s, y, D_y)^* \Delta \psi(s, D_s, y, D_y)$$

where $D_{x_j} := \frac{\partial}{i\partial x_j}$ and where $\psi(s, \sigma, y, \eta)$ is supported in (0.1) and identically one in some smaller conic neighborhood. Often we omit explicit mention of the microlocal cut-off ψ in calculations which are valid on its microsupport. Under the exponential map

$$\exp : N_\gamma \rightarrow M$$

along the normal bundle to γ , the localization of g , resp. Δ , to the tubular neighborhood $|y| < \epsilon$ pulls back to a locally well-defined metric, respectively Laplacian, on a similar neighborhood in N_γ . Hence \exp conjugates Δ_ψ to an isometric microlocalized Laplacian on N_γ , which we continue to denote by Δ_ψ . We are thus reduced to calculating the wave invariants of a Laplacian on the model space $S_L^1 \times \mathbb{R}^n$ at the closed geodesic $\gamma = S_L^1 \times \{0\}$, where $S_L^1 := \mathbb{R}/L\mathbb{Z}$. For the sake of simplicity we assume the normal bundle is orientable but note that the reduction is valid for immersed as well as embedded closed geodesics.

We now wish to put Δ_ψ into normal form, which is first of all to conjugate it (modulo a small error) into a distinguished maximal abelian algebra \mathcal{A} of pseudodifferential operators on the model space $S_L^1 \times \mathbb{R}^n$. Roughly speaking, \mathcal{A} is generated by the tangential operator $D_s := \frac{\partial}{i\partial s}$ on S_L^1 together with the transverse harmonic oscillators

$$(0.2) \quad I_j = I_j(y, D_y) := \frac{1}{2}(D_{y_j}^2 + y_j^2).$$

In the construction of the normal form, a special role will be played by the distinguished element

$$(0.3) \quad \mathcal{R} := \frac{1}{L}(LD_s + H_\alpha)$$

where

$$(0.4) \quad H_\alpha := \frac{1}{2} \sum_{k=1}^n \alpha_k I_k$$

and where the choice of sign in $\pm \alpha_k$ will be specified below. This element comes up naturally as the semi-classical parameter in the construction of quasi-modes, although D_s is more suitable for analysing the wave invariants. Note that both are elliptic elements in the conic neighborhood

$$(0.5) \quad |I_j| < \epsilon \sigma \quad I_j(y, \eta) := \frac{1}{2}(y_j^2 + \eta_j^2),$$

which will be the image of (0.1) under the conjugation to normal form.

The classical Birkhoff normal form theorem states roughly the following: near a non-degenerate elliptic closed geodesic γ the Hamiltonian

$$H(x, \xi) = |\xi| := \sqrt{\sum_{ij=1}^{n+1} g^{ij} \xi_i \xi_j}$$

can be conjugated by a homogeneous local canonical transformation χ to the normal form

$$(0.6) \quad \chi^* H \equiv \sigma + \frac{1}{L} \sum_{i,j=1}^n \alpha_j I_j + \frac{p_1(I_1, \dots, I_n)}{\sigma} + \dots \quad \text{mod} \quad O_\infty^1$$

where p_k is homogeneous of order $k+1$ in I_1, \dots, I_n , and where O_∞^1 is the space of germs of functions homogeneous of degree 1 which vanish to infinite order along γ . Note that all the terms in (0.6) are homogenous of degree 1 in $(\sigma, I_1, \dots, I_n)$, and that the order of vanishing at $|I| = 0$ equals one plus the order of decay in σ . The coefficients of the monomials in the $p_j(I_1, \dots, I_n)$ are known as the classical Birkhoff normal form invariants. (See the Appendix for some further details).

The quantum Birkhoff normal form is the more or less analogous statement on the operator level. In the following the symbol \equiv means that the two sides agree modulo operators whose complete symbols are of order 1 and vanish to infinite order on γ . Also, $O_m \Psi^r$ denotes the space of pseudodifferential operators of order r whose complete symbols vanish to order m at $(y, \eta) = (0, 0)$.

Theorem B *There exists a microlocally elliptic Fourier Integral operator W from the conic neighborhood (0.1) of $\mathbb{R}^+ \gamma$ in $T^*N_\gamma - 0$ to the conic neighborhood (0.5) of $T_+^* S_L^1$ in $T^*(S_L^1 \times \mathbb{R}^n)$ such that:*

$$\begin{aligned} \mathcal{D} &:= W \sqrt{\Delta_\psi} W^{-1} \equiv \bar{\psi}(\mathcal{R}, I_1, \dots, I_n) \left[\mathcal{R} + \frac{p_1(I_1, \dots, I_n)}{L\mathcal{R}} + \frac{p_2(I_1, \dots, I_n)}{(L\mathcal{R})^2} + \dots + \frac{p_{k+1}(I_1, \dots, I_n)}{(L\mathcal{R})^{k+1}} + \dots \right] \\ &\equiv D_s + \frac{1}{L} H_\alpha + \frac{\tilde{p}_1(I_1, \dots, I_n)}{LD_s} + \frac{\tilde{p}_2(I_1, \dots, I_n)}{(LD_s)^2} + \dots + \frac{\tilde{p}_{k+1}(I_1, \dots, I_n)}{(LD_s)^{k+1}} + \dots \end{aligned}$$

where the numerators $p_j(I_1, \dots, I_n), \tilde{p}_j(I_1, \dots, I_n)$ are polynomials of degree $j+1$ in the variables I_1, \dots, I_n , where $\bar{\psi}$ is microlocally supported in (0.5), and where W^{-1} denotes a microlocal inverse to W in (0.5). The k th remainder term lies in the space $\oplus_{j=0}^{k+2} O_{2(k+2-j)} \Psi^{1-j}$.

The QBNF coefficients will by definition be the coefficients of the monomials in the classical action (I_j -) variables in the complete Weyl symbols of the operators $\tilde{p}_j(I_1, \dots, I_n)$. As mentioned above, the proof of Theorem B gives an effective method for calculating them as integrals over γ of FJ polynomials.

The asymptotic relation in the above expansion may be viewed in either of two ways: First, as mentioned in the statement of the Theorem, the k th remainder is a sum of terms in \mathcal{A} of orders $1, 0, \dots, -(k+1)$ where the complete symbol of the term of order $1-j$ must vanish to order $2(k+2-j)$. This characterization of the remainder will play the key role in the calculation of the wave invariants, since terms in the normal form with low pseudodifferential order or with high vanishing order make no contribution to a given wave invariant. On the other hand, it may be viewed as a semi-classical asymptotic relation with \mathcal{R} playing the role of semi-classical parameter; thus the theorem gives a semi-classical expansion for \mathcal{D} in terms of \mathcal{R} . This point of view comes up naturally in the theory of quasi-modes associated to γ : Indeed, consider the joint \mathcal{A} - eigenfunctions

$$(0.7) \quad \phi_{kq}^o(s, y) := e_k(s) \otimes \gamma_q(y)$$

with $e_k(s) := e^{\frac{2\pi}{L} i k s}$ and with γ_q q th normalized Hermite function ($q \in \mathbb{N}^n$). The corresponding eigenvalues of \mathcal{D} then have the semi-classical expansions

$$(0.8) \quad \lambda_{kq} \equiv r_{kq} + \frac{p_1(q)}{r_{kq}} + \frac{p_2(q)}{r_{kq}^2} + \dots$$

where

$$(0.9) \quad r_{kq} = \frac{1}{L} (2\pi k + \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j)$$

are the eigenvalues of \mathcal{R} . Here the index q is held fixed as $k \rightarrow \infty$. We recognize in (0.8) the familiar form of the quasi-eigenvalues associated to γ (cf. [B.B., ch.9]); hence the intertwining operator W is the operator taking the eigenfunctions (0.7) to quasi-modes of infinite order for Δ at γ .

As will be seen in (§4), Theorem B implies that the wave invariants of $\sqrt{\Delta_\psi}$ are the same as the wave invariants of \mathcal{D} . The second main step in the calculation of the wave invariants is then the use of the non-commutative residue to connect the terms in the normal form expansion with the terms in the singularity expansion for $Tr \exp it\mathcal{D}$. The main point here is that

$$(0.10) \quad a_{\gamma k} = res D_t^k e^{it\sqrt{\Delta_\psi}} := Res_{s=0} Tr D_t^k e^{it\sqrt{\Delta_\psi}} \sqrt{\Delta_\psi}^{-s},$$

with res invariant under conjugation by (microlocal) unitary operators, and depending on only a finite jet of the Laplacian near γ . Hence it may be calculated by conjugating to the normal form, and indeed will only depend on a finite part of the normal form. Applying D_t^k and formally exponentiating the terms of order ≤ -1 in D_s we get

$$(0.11) \quad res \bar{\psi}(D_s, I_1, \dots, I_n) D_t^k e^{it\mathcal{D}}|_{t=L} = res \bar{\psi}(D_s, I_1, \dots, I_n) e^{iLD_s} e^{iH_\alpha} \mathcal{D}^k (I + iL \frac{\tilde{p}_1(I_1, \dots, I_n)}{LD_s} + \dots)$$

which suggests that the wave coefficient $a_{k\gamma}$ is the regularized trace of the coefficient of D_s^{-1} in (0.11). This is not clear, even formally, since many of the terms of negative order in D_s in the exponent have overall order 1 as pseudodifferential operators; but it will prove to be the case. Since $e^{iLD_s} = I$ on $\mathbb{R}/L\mathbb{Z}$ the Fourier Integral factor in (0.11) is just e^{iH_α} . Regarding the regularized traces of the coefficients, we note that

$$(0.12) \quad Tre^{iH_\alpha} = \sum_{q \in \mathbb{N}^n} e^{i \sum_{k=1}^n (q_k + \frac{1}{2}) \alpha_k}$$

is well-defined as the tempered distribution

$$(0.13) \quad T(\alpha) = \prod_{k=1}^n \frac{e^{\frac{i}{2}(\alpha_k + i0)}}{(1 - e^{i(\alpha_k + i0)})}$$

on \mathbb{R}_α^n . Since its singular support is the union of the hyperplanes $\mathbb{Z}_{km} := \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \alpha_k = 2\pi m\}$, T has smooth localization to neighborhoods of Floquet exponents $(\alpha_1, \dots, \alpha_n)$ which are independent of π over \mathbb{Q} . Hence the regularized trace is simply the evaluation of the distribution trace at a regular point. Similarly, the coefficient of D_s^{-k-1} in (0.12) has a distribution trace of the form

$$(0.14) \quad \sum_{q \in \mathbb{N}^n} \mathcal{F}_{k,-1}(q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}) e^{i \sum_{k=1}^n (q_k + \frac{1}{2}) \alpha_k} \\ = \mathcal{F}_{k,-1}(D_{\alpha_1}, \dots, D_{\alpha_n}) T(\alpha)$$

for a certain polynomial $\mathcal{F}_{k,-1}$. Hence this trace is also a locally smooth function in neighborhoods of non-resonant exponents.

Theorem C *The wave invariants are given by*

$$a_{k\gamma} = \mathcal{F}_{k,-1}(D_{\alpha_1}, \dots, D_{\alpha_n}) T(\alpha).$$

The coefficients of the polynomials $\mathcal{F}_{k,-1}$ are evidently polynomials in the QBNF coefficients and the differentiation process produces polynomials in the β_j 's. Combined with Theorem B and a dimensional analysis, this proves Theorem A.

The proofs of Theorems A-C also lead to a somewhat simpler proof of Guillemin's inverse theorem that the classical (in fact the full quantum) normal form is determined by the wave trace invariants for all the iterates of γ . We only sketch the proof here, assuming the reader's familiarity with the original proof of Guillemin in [G.2]. The key point is to focus on the Floquet invariants $\beta_j := (1 - e^{\theta_j})^{-1}$ for all the iterates

γ^m of γ , that is the residues (0.10) for $t = L, 2L, 3L, \dots$. It follows from the calculations in Theorems A-C that the wave invariants are polynomials in the β_j 's: more precisely, for each m , $a_{k\gamma^m}$ is the special value at $\theta_j = m\alpha_j$ of the fixed polynomial $I_{\gamma;k}$ in the variables β_j . Under the irrationality condition above, the points $(e^{im\alpha_1}, \dots, e^{im\alpha_n})$ form a dense set on the torus, and hence the special values at these points determine the entire polynomial. The coefficients of the k th polynomial $I_{\gamma;k}$ are therefore determined by the wave invariants for γ, γ^2, \dots . By studying the relation of the coefficients of $I_{\gamma;k}$ to the normal form invariants, Guillemin proves that all of the latter can be determined from the former.

Although Theorems A-C are only proved here under the hypothesis that γ is non-degenerate elliptic, they have analogues for hyperbolic and mixed hyperbolic-elliptic geodesics, which we plan to describe in a future article [Z.3]. We also note that for closed geodesics possessing neighborhoods in which the metric has no pairs of conjugate points for $t \leq L$, the wave invariants can be calculated directly from a Hadamard parametrix [D] [Z.1,5]. Since a sufficiently large number of iterates of an elliptic closed geodesics will always contain pairs of conjugate points, but a small iterate may contain none, the calculation here and in [Z.1] overlap but are independent. The calculations of [Z.1] also apply to hyperbolic geodesics without pairs of conjugate points, showing that the form of the wave invariants is essentially the same for the hyperbolic and elliptic cases. However the form resulting from the Hadamard parametrix is not immediately that of FJF polynomials, and it takes considerable manipulation to show that the formulae given here and in that paper agree. In the opposite extreme of Zoll manifolds, all of whose geodesics are closed and of completely degenerate elliptic type, the wave invariants are calculated in [Z3,4] by yet another method.

The organization of this paper is as follows:

- §1: The models
- §2: Semi-classical normal form of the Laplacian
- §3: Normal form: Proof of Theorem B
- §4: Residues and wave invariants: Proof of Theorem C
- §5: Local formulae for the residues: Proof of Theorem A
- §6: Quantum Birkhoff normal form coefficients
- §7: Explicit formulae in dimension 2
- §8: Appendix: The classical Birkhoff normal form
- §9: Index of Notation

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1. THE MODELS

As mentioned above, the calculation of the wave invariants associated to a closed geodesic γ of a Riemannian manifold (M, g) can be transplanted to the normal bundle N_γ by means of the exponential map. Thus, the model space is the cylinder $S_L^1 \times \mathbb{R}^n$, where as above $S_L^1 = \mathbb{R}/L\mathbb{Z}$. In this section we first collect together some basic formulae and facts concerning the ‘‘Hermite package’’ on this model space: that is, those aspects of analysis which come from the representation theory of the Heisenberg and metaplectic algebras. We then transfer the Hermite package to N_γ in a way particularly well-adapted to the metric along γ .

§1.1: The model: $\mathcal{H} = H^2(S_L^1) \otimes L^2(\mathbb{R}^n)$

Since we are only concerned with the conic neighborhood (0.2) of $\mathbb{R}^+\gamma$, we only consider the positive part $T_+^*S_L^1 \times T^*\mathbb{R}^n$ and its quantum analogue the Hardy space $H^2(S_L^1) \otimes L^2(\mathbb{R}^n)$ with $k \geq 0$.

On the phase space level, the model is roughly $T^*(S_L^1 \times \mathbb{R}^n)$. More precisely it is the cone (0.1) in the natural symplectic coordinates (s, σ, y, η) . Since (0.1) is a conic neighborhood of $\mathbb{R}^+\gamma$, we will view it as a subcone of the positive part $T_+^*S_L^1 \times T^*\mathbb{R}^n$ ($\sigma > 0$). On the Hilbert space level the model is then

$\mathcal{H} := H^2(S_L^1) \otimes L^2(\mathbb{R}^n)$, where $H^2(S_L^1)$ is the Hardy space, or more precisely the range \mathcal{H}_ψ of the microlocal cutoff ψ of the introduction; generally we omit the subscript unless we need to emphasize the role of ψ . We now introduce some distinguished algebras of operators on the model space.

First is the (complexified) Heisenberg algebra $\mathbf{h}_n \otimes \mathbb{C}$, which will be identified with its usual realization on $L^2(\mathbb{R}^n)$. It is then generated by the elements $y_j = \text{“multiplication by } y_j\text{”}$ and by $D_{y_j} = \frac{\partial}{i\partial y_j}$, or equivalently by the annihilation, resp. creation, operators

$$A_j := y_j + iD_{y_j} \quad A_j^* = y_j - iD_{y_j}$$

which satisfy the commutation relations

$$[A_j, A_k] = [A_j^*, A_k^*] = 0 \quad [A_j, A_k^*] = 2\delta_{ij}I.$$

The enveloping algebra of the Heisenberg algebra

$$(1.1.1) \quad \mathcal{E} := \langle Y_1, \dots, Y_n, D_1, \dots, D_n \rangle$$

is the algebra of partial differential operators on \mathbb{R}^n with polynomial coefficients. We let \mathcal{E}^n denote the subspace of polynomials of degree n in the variables y_j, D_{y_j} . In the usual isotropic Weyl algebra \mathcal{W}^* of pseudo-differential operators on \mathbb{R}^n , the operators y_j, D_{y_j} are given the order $\frac{1}{2}$, so that

$$(1.1.1b) \quad \begin{aligned} \mathcal{E}^n &\subset \mathcal{W}^{n/2} \\ [\mathcal{E}^m, \mathcal{E}^n] &\subset \mathcal{E}^{m+n-2}. \end{aligned}$$

The symplectic algebra $sp(n, \mathbb{C})$ is represented in \mathcal{E}^2 by homogeneous quadratic polynomials in Y_j, D_j , and a maximal abelian subalgebra of it is spanned by the harmonic oscillators (0.2). We denote by

$$(1.1.2a) \quad \mathcal{I} := \langle I_1, \dots, I_n \rangle$$

the (maximal abelian) subalgebra they generate in \mathcal{W} , with $\mathcal{I}^k := \mathcal{I} \cap \mathcal{W}^k$, and by

$$(1.1.2b) \quad \mathcal{P}_{\mathcal{I}} = \mathcal{I} \cap \mathcal{E}$$

the subalgebra of polynomials in the generators (0.2), with $\mathcal{P}_{\mathcal{I}}^k$ the space of polynomials of degree k .

The full pseudo-differential algebra on $S_L^1 \times \mathbb{R}^n$ is the doubly filtered algebra

$$\Psi^{**}(S_L^1 \times \mathbb{R}^n) \equiv \Psi^*(S_L^1) \otimes \mathcal{W}^*$$

with

$$\Psi^{mn}(S_L^1 \times \mathbb{R}^n) \equiv \Psi^m(S_L^1) \otimes \mathcal{W}^n.$$

A maximal abelian subalgebra of it is given by

$$(1.1.3) \quad \mathcal{A} := \langle D_s, I_1, \dots, I_n \rangle = \langle \mathcal{R}, I_1, \dots, I_n \rangle$$

where \mathcal{R} is the distinguished element (0.3). It inherits a double filtration \mathcal{A}^{mn} . As above, our interest is really in the microlocalization of (1.1.3) to the cone (0.2), i.e. the operators in (1.1.3) will only be used in composition with the microlocal cut-off $\overline{\psi}(\mathcal{R}, I_1, \dots, I_n)$. In this cone D_s and \mathcal{R} are elliptic. Hence the subalgebra

$$(1.1.4) \quad \langle \mathcal{R} \rangle \otimes \mathcal{P}_{\mathcal{I}}$$

of pseudo-differential symbols in \mathcal{R} with coefficients in $\mathcal{P}_{\mathcal{I}}$ is well defined.

An orthonormal basis of $L^2(\mathbb{R}^n)$ of joint eigenfunctions of \mathcal{I} is provided by the Hermite functions $\gamma_q, q \in \mathbb{N}^n$. Here, γ_o is the Gaussian $\gamma_o(y) = \gamma_{iI}(y) := e^{-\frac{1}{2}|y|^2}$. It is the unique “vacuum state”, i.e. the state annihilated by the annihilation operators. The q th Hermite function is then given by $\gamma_q := C_q A_1^{*q_1} \dots A_n^{*q_n} \gamma_o (q \in \mathbb{N}^n)$, with $C_q = (2\pi)^{-n/2} (q!)^{-1/2}, q! = q_1! \dots q_n!$. The notation “ γ_q ” for “Gaussian” is standard in this context, see [F], and should not be confused with the notation for closed geodesics. An orthonormal basis of $H^2(S_L^1) \otimes L^2(\mathbb{R}^n)$ of joint \mathcal{A} -eigenfunctions is then furnished by

$$(1.1.5) \quad \phi_{kq}^o(s, y) := e_k(s) \otimes \gamma_q(y), \quad e_k(s) := e^{\frac{2\pi}{L}iks}.$$

§1.2: The twisted model \mathcal{H}_α

We now introduce a unitarily equivalent (twisted) version of the model, in which the distinguished element \mathcal{R} gets conjugated to D_s . This will eventually help to simplify the transport equations in §2.

The unitary equivalence will be given by conjugation with the unitary operator

$$(1.2.1a) \quad \mu(r_\alpha) := \int_{S_L^1}^{\oplus} \mu(r_\alpha(s)) ds = \int_{S_L^1}^{\oplus} e^{i \frac{s}{L} H_\alpha} ds$$

where μ is the metaplectic representation, and where $r_\alpha(s)$ is the block diagonal orthogonal transformation on \mathbb{R}^{2n} with blocks

$$(1.2.1b) \quad r_{\alpha_j}(s) := \begin{pmatrix} \cos \alpha_j \frac{s}{L} & \sin \alpha_j \frac{s}{L} \\ -\sin \alpha_j \frac{s}{L} & \cos \alpha_j \frac{s}{L} \end{pmatrix}.$$

The direct integral here refers to the representation of $L^2(S^1) \otimes L^2(\mathbb{R}^n)$ as $\int_{S_L^1} L^2(\mathbb{R}^n) ds$, that is,

$$\int_{S_L^1}^{\oplus} \mu(r_\alpha(s)) ds f(s, y) = \mu(r_\alpha(s)) f(s, y)$$

where the right side is the application of the operator in the y -variables with s fixed. As will be seen below, $\mu(r_\alpha)$ conjugates \mathcal{R} to D_s , and commutes with I_1, \dots, I_n . Hence it preserves the algebra (1.1.3).

On the other hand, it does not preserve the Hilbert space \mathcal{H} . Indeed, the elements ϕ_{kq}^o get transformed into the elements

$$(1.2.2) \quad e_{kq}(s) \otimes \gamma_q(y), \quad e_{kq}(s) := e^{i r_{kq} s}$$

which are not periodic in s . Rather they satisfy

$$(1.2.3a) \quad e_{kq} \otimes \gamma(s + L, y) = e^{i \kappa_q} e_{kq} \otimes \gamma(s, y)$$

with

$$(1.2.3b) \quad \kappa_q = \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j.$$

The space $H^2(S_L^1) \otimes L^2(\mathbb{R}^n)$ thus gets taken to the space \mathcal{H}_α of elements of the form

$$(1.2.4) \quad f(s, y) = \sum_{k=o}^{\infty} \sum_{q \in \mathbb{N}^n} \hat{f}(k, q) e_{kq} \otimes \gamma_q$$

with square summable coefficients.

We can better describe this Hilbert space (and its associated phase space) in the language of ‘quantized mapping cylinders’.

On the phase space level we have the symplectic map $r_\alpha(L)$ of $T^*\mathbb{R}^n$ of (1.2.1b), essentially the Poincaré map of our problem. As in [F.G][G.2] we can introduce its (homogeneous) symplectic mapping cylinder $C_{r_\alpha(L)}$: namely, the quotient of $T^*\mathbb{R} \times T^*\mathbb{R}^n$ under the cyclic group $\langle R_\alpha(L) \rangle$ generated by the symplectic map

$$(1.2.5) \quad \tilde{r}_\alpha(L) : T^*\mathbb{R} \times T^*\mathbb{R}^n \rightarrow T^*\mathbb{R} \times T^*\mathbb{R}^n, \quad \tilde{r}_\alpha(L)(s, \sigma, y, \eta) := (s + L, \sigma, r_\alpha(L)(y, \eta)).$$

Note that the first return time is constant, which is consistent with [F.G][G.2, 2.10] since elements of $Sp(2n, \mathbb{R})$ preserve the contact form $ds - \frac{1}{2}(\eta dy - y d\eta)$. Also note that the mapping cylinder can be untwisted via the symplectic map

$$(1.2.7) \quad R_\alpha : T^*\mathbb{R} \times T^*\mathbb{R}^n \rightarrow T^*\mathbb{R} \times T^*\mathbb{R}^n, \quad R_\alpha(s, \sigma, y, \eta) := (s, \sigma + \frac{1}{2} \sum_j \alpha_j (y_j^2 + \eta_j^2), r_\alpha(s)(y, \eta)).$$

Indeed, we have

$$(1.2.8) \quad R_\alpha(s + L, \sigma, y, \eta) = \tilde{r}_\alpha(L) R_\alpha(s, \sigma, y, \eta)$$

so that R_α induces a symplectic equivalence $C_{r_\alpha(L)} \sim T^*(S_L^1) \times R^*(\mathbb{R}^n)$.

On the quantum level, the analogue of the mapping cylinder is the Hilbert space \mathcal{H}_α of functions on $\mathbb{R} \times \mathbb{R}^n$ satisfying

$$(1.2.6) \quad f(s+L, y) = \mu(r_\alpha(L))f(s, y)$$

and square integrable on $[0, L) \times \mathbb{R}^n$. The intertwining operator $\mu(r_\alpha)$ is essentially the analogue of R_α . More precisely, we have:

(1.2.9) Proposition

- (i) $\mu(r_\alpha)^* D_s \mu(r_\alpha) = \mathcal{R}$;
- (ii) $\mu(r_\alpha) \phi_{kq}^o(s, y) = e^{i r_{kq} s} \gamma_q$;
- (iii) $\mu(r_\alpha) : \mathcal{H} \rightarrow \mathcal{H}_\alpha$.

Proof:

- (i) Follows from the fact that μ takes the j th diagonal block

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to I_j and hence that $\mu(r_\alpha(s)) = e^{i \frac{s}{L} H_\alpha}$.

- (ii) Follows from (i) and the fact that γ_q is an eigenfunction of eigenvalue $q_j + \frac{1}{2}$ of I_j .

- (iii) Follows from (ii). □

Under conjugation by $\mu(r_\alpha)$, the sub-algebra (1.1.4) goes over to the algebra

$$(1.2.10) \quad \langle D_s \rangle \otimes \mathcal{P}_I$$

of pseudodifferential symbols in D_s with coefficients in the I_j 's. As usual, it is understood to be microlocalized to the cone (0.2).

We now conjugate by a further unitary equivalence to transfer this Hermite package to N_γ . It will induce a new model better adapted to the geometry of (M, g) near γ ; we call it the adapted model.

§1.3: The adapted model

To define it, we must discuss the Jacobi equation along γ . Let $Y(s)$ be a vector field along γ , and as above let $Y(s) = \sum_{i=1}^n Y_i(s) e_i(s)$ be its expression in terms of the parallel normal frame. The Jacobi equation is then

$$\frac{d^2}{ds^2} Y_i + \sum_{j=1}^n K_{ij} Y_j = 0.$$

Let $\mathcal{J}_\gamma^\perp \otimes \mathbf{C}$ denote the space of complex orthogonal Jacobi fields along γ . Equipped with the symplectic form $\omega(X, Y) := \langle X, \frac{DY}{ds} \rangle - \langle \frac{DX}{ds}, Y \rangle$, it is a symplectic vector space of dimension $2n$ (over \mathbf{C}). Here $\frac{D}{ds}$ denotes covariant differentiation along γ , and \langle, \rangle is the inner product defined by the metric. Now let $Z(s) := (Y(s), \frac{DY}{ds})$. The linear Poincare map P_γ is by definition the operator on $\mathcal{J}_\gamma^\perp \otimes \mathbf{C}$ given by $P_\gamma Z(t) = Z(t+L)$. We recall that γ is assumed non-degenerate elliptic, hence

$$\text{Spec}(P_\gamma) \subset S^1 \quad \pm 1 \notin \text{spec}(P_\gamma).$$

Since $P_\gamma \in Sp(2n, \mathbf{R})$, its eigenvalues come in complex conjugate pairs $\{e^{i\alpha_k}, e^{-i\alpha_k}\}$. The exponents α_k (called Floquet) are only defined up multiples of 2π ; they will be normalized below as in [B.B, (9.3.17)]. Then there exists a basis of complex eigenvectors $\{Y_1, \dots, Y_n\}$ satisfying

$$(1.3.1) \quad P_\gamma Y_k = e^{i\alpha_k} Y_k \quad Y_j \in \mathcal{J} \otimes \mathbf{C}$$

$$\omega(Y_k, Y_j) = \omega(\overline{Y}_j, \overline{Y}_k) = 0, \quad \omega(Y_j, \overline{Y}_k) = \delta_{jk}.$$

with one choice of eigenvalue from each complex conjugate pair. Equivalently, the span of $\{Y_1, \dots, Y_n\}$ defines a P_γ -invariant positive Lagrangean subspace of $\mathcal{J}_\gamma^\perp \otimes \mathbf{C}$.

Consider now the (modified) Wronskian matrix

$$(1.3.2) \quad a_s := \begin{pmatrix} \text{Im} \dot{Y}(s)^* & \text{Im} Y(s)^* \\ \text{Re} \dot{Y}(s)^* & \text{Re} Y(s)^* \end{pmatrix}.$$

Here, with a little abuse of notation, we denote the components of $Y(s)$, resp. $\frac{DY}{ds}$ relative to the parallel normal frame $e(s)$ by $Y(s)$, resp. $\dot{Y}(s)$ and the notation A^* refers to the adjoint of a matrix A . From (1.3.1) we have that $a_s \in Sp(2n, \mathbf{R})$, which is equivalent to

$$\overline{Y} \frac{DY}{ds} - \frac{\overline{DY}}{ds} Y = iI.$$

As above, we let μ denote the metaplectic representation. Identifying a_s with one of its two possible lifts from $Sp(2n, \mathbf{R})$ to $Mp(2n, \mathbf{R})$ we introduce the unitary operator

$$(1.3.3) \quad \mu(a) := \int_{\gamma}^{\oplus} \mu(a_s) ds$$

on $\int_{S_L^1}^{\oplus} L^2(\mathbf{R}^n) ds$. In other words,

$$\mu(a)f(s, y) = \mu(a_s)f(s, y)$$

where the operator on the right side acts in the y -variables. Informally, we think of the range as the Hilbert space $\int_{\gamma}^{\oplus} L^2(N_{\gamma(s)}) ds$, and below we will describe it more completely in terms of quantum mapping cylinders. We now use $\mu(a)$ to transfer the Hermite package to N_{γ} . We begin with the generators of the Heisenberg algebra, and set:

$$\begin{aligned} P_j &:= \mu(a)^* D_j \mu(a) & Q_j &:= \mu(a)^* Y_j \mu(a) \\ \Lambda_j &:= \mu(a)^* A_j \mu(a) & \Lambda_j^* &:= \mu(a)^* A_j^* \mu(a) \end{aligned} \quad .$$

$$I_{\gamma j} := \frac{1}{2} \Lambda_j \Lambda_j^*.$$

We will refer to the Λ_j 's, resp. Λ_j^* 's, as the adapted annihilation and creation operators and to the operators $I_{\gamma j}$ in the symplectic algebra as the adapted action operators. These adapted operators play a key role in the study of quasi-modes associated to γ . To establish the connection, we now verify that they coincide with the operators similarly denoted in [B.B., ch.9].

First, some remarks on notation. For reasons that will become clearer below (§1.4-5), we change the notation for the transverse coordinates from y to u , which should be thought of as the rescaled coordinates $u = L^{-\frac{1}{2}}y$. Objects in the adapted model will henceforth always be expressed in terms of u -coordinates. For instance, multiplication by u_j will be denoted simply by u_j and differentiation in u_j by D_{u_j} . Also, to quote easily some basic facts about the metaplectic representation from Folland [F], we will conform to the following 'transposed' notation for the remainder of this section: symbols of operators will be denoted $\sigma(p, q)$ rather than $\sigma(q, p)$, and the corresponding Weyl pseudodifferential operator will be denoted by $\sigma(D, x)$ (later we will also use the more standard notation $\sigma^w(x, D)$). An element of $Sp(n, \mathbf{R})$ will be denoted by \mathcal{A} and its transpose by \mathcal{A}^* .

(1.3.4) **Proposition**

$$\Lambda_j = \sum_{k=1}^n (iy_{jk} D_{u_k} - \frac{dy_{jk}}{ds} u_k) \quad \Lambda_j^* = \sum_{k=1}^n (-i\overline{y}_{jk} D_{u_k} - \overline{\frac{dy_{jk}}{ds}} u_k)$$

Proof: This follows from the metaplectic covariance of the Weyl calculus, that is from the identity [F, Theorem 2.15]

$$(\sigma \cdot \mathcal{A})(D, x) = \mu(\mathcal{A}^*) \sigma(D, x) \mu(\mathcal{A}^*)^{-1}.$$

If we set $a = \mathcal{A}^*$ we get (in an obvious vector notation)

$$\mu(a)^*(u_j + iD_{u_j})\mu(a) = i(Y_j \cdot D_u) - \dot{Y}_j \cdot u$$

(compare [BB, ch.9]). □

As for the tangential operator, we have:

(1.3.5) **Proposition** *The image \mathcal{L} of D_s under μ is given by:*

$$\mathcal{L} := \mu(a)^* D_s \mu(a) = D_s - \frac{1}{2} \left(\sum_{j=1}^n D_{u_j}^2 + \sum_{ij=1}^n K_{ij}(s) u_i u_j \right).$$

Proof: The left side is equal to $(D_s + \mu(a_s)^* D_s \mu(a_s))$. To evaluate the second term, we use that both $\text{Re}Y(s)$ and $\text{Im}Y(s)$ are Jacobi fields, and that Jacobi's equation is equivalent to the linear system $\frac{D}{ds}(Y, P) = JH(Y, P)$. Here, $P = \frac{DY}{ds}$, J is the standard complex structure on R^{2n} , and

$$H = \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}$$

where K is the curvature matrix and I is the identity matrix [B.B, (9.2.9)]. Hence, the second term is $\frac{1}{2} d\mu(JH)$ with $d\mu$ the derived metaplectic representation. But $\frac{1}{2} d\mu(JH) = 1/2 (\sum_{i=1}^n \partial_{u_i}^2 - \sum_{ij=1}^n K_{ij}(s) u_i u_j)$ [F]. \square

We now consider the appropriate Hilbert space (quantized mapping cylinder) in the adapted model. We first note:

(1.3.6) **Proposition**

(i)

$$\mu(a^{-1}) \gamma_o(s, u) := U_o(s, u) = (\det Y(s))^{-1/2} e^{i \frac{1}{2} \langle \Gamma(s) u, u \rangle}$$

where $\Gamma(s) := \frac{dY}{ds} Y^{-1}$.

(ii)

$$\mu(a^{-1}) \gamma_q := U_q = \Lambda_1^{q_1} \dots \Lambda_n^{q_n} U_o.$$

Proof: Let us recall the action of the metaplectic representation on Gaussians [F, Ch. 4.5]. For Z in the Siegel upper half space ($n \times n$ complex matrices $Z = X + iY$ with $Y >> 0$), we have the Gaussian

$$\gamma_Z(x) := e^{i \frac{1}{2} \langle Zx, x \rangle}$$

on \mathbb{R}^n . The action of an element $\mathcal{A} \in Mp(2n, \mathbb{R})$ on the Gaussian is given by

$$\mu(\mathcal{A}^{*-1}) \gamma_Z = m(\mathcal{A}, Z) \gamma_{\alpha(\mathcal{A})Z}$$

where

$$m(\mathcal{A}, Z) = \det^{-1/2}(CZ + D), \alpha(\mathcal{A})Z = (AZ + B)(CZ + D)^{-1}$$

for

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(see [F, 4.65].) Writing

$$Y(s) = \text{Im}Y(s)i + \text{Re}Y(s), \quad \Gamma(s) = (\text{Im}\dot{Y}(s)i + \text{Re}\dot{Y}(s))(\text{Im}Y(s)i + \text{Re}Y(s))^{-1}$$

we see that $m(\mathcal{A}, iI) \gamma_{\alpha(\mathcal{A})iI}(u) = (\det Y(s))^{-1/2} \exp(i \frac{1}{2} \langle \Gamma(s) u, u \rangle)$ if

$$\mathcal{A} = \begin{pmatrix} \text{Im}\dot{Y}(s) & \text{Re}\dot{Y}(s) \\ \text{Im}Y(s) & \text{Re}Y(s) \end{pmatrix}.$$

The formula (i) follows from $\mathcal{A} = a^*$ and (ii) is an immediate consequence of (i). \square

Consider now the periodicity properties of the above data under $s \rightarrow s + L$. We observe that a_s fails to be periodic in s for two reasons: first, due to the holonomy of the frame $e(s)$, and secondly due to the monodromy of the Jacobi fields $Y(s)$. Indeed, we have:

$$(1.3.7a) \quad a_{(s+L)} = T a_s r_\alpha^{-1}(L)$$

with r_α as in §1.2 and where T , the holonomy matrix, is the $2n$ by $2n$ block diagonal matrix with equal diagonal blocks $t = (t_{ij})$ satisfying

$$e_i(L) = \sum_{j=1}^n t_{ij} e_j(0).$$

It is of course the lift to $T^*\mathbb{R}^n$ of a rotation on the base. The two properties can be summarized by writing

$$(1.3.7b) \quad \tilde{a}_{s+L} = T\tilde{a}_s$$

with $\tilde{a}_s := a_s r_\alpha(s)$.

As in §1.2 we will reformulate (1.3.7a-b) in terms of quantum mapping cylinders. First, we put

$$(1.3.8) \quad C_T^\infty(\mathbb{R} \times \mathbb{R}^n) := \{f \in C^\infty(\mathbb{R} \times \mathbb{R}^n) : f(s+L, u) = \mu(T)f(s, u)\}$$

and let \mathcal{H}_T denote its closure with respect to the obvious inner product over $[0, L) \times \mathbb{R}^n$. Note that the metaplectic operator $\mu(T)$ is simply

$$\mu(T)f(u) = f(t^{-1}u)$$

and hence that

$$C_T^\infty(\mathbb{R} \times \mathbb{R}^n) \sim C^\infty(N_\gamma)$$

where the isomorphism is simply the pull-back by the exponential map defined by the frame $e(s)$. Thus:

(1.3.8) **Proposition**

(i) Let $P(s, D_s, u, D_u)$ be a partial differential operator on N_γ , expressed in the coordinates (s, u) . Then

$$P(s+L, D_s, u, D_u) = \mu(T)P(s, D_s, u, D_u)\mu(T)^*;$$

(ii) The functions

$$\mu(\tilde{a}_s)(\phi_{kq}^o) = \phi_{kq} := e^{ir_{kq}s} U_q(s, u)$$

define a smooth orthonormal basis of \mathcal{H}_T .

Proof

(i) It suffices to prove this when P is a vector field given in the local normal coordinates by $a_o(s, u)D_s + \sum_{j=1}^n a_j(s, u)D_{u_j}$. Since the metaplectic operator $\mu(T)$ corresponding to T is the operator $f(u) \rightarrow f(t^{-1}u)$, we have

$$\mu(T)P\mu(T)^* = a_o(s, t^{-1}u)D_s + \sum_{j=1}^n a_j(s, t^{-1}u)t_{jk}D_{u_k}.$$

On the other hand, the vector field is well-defined on N_γ if and only if

$$a_o(s+L, t^{-1}u)D_s + \sum a_i(s+L, t^{-1}u)t_{ij}D_{u_j} = a_o(s, u)D_s + \sum a_j(s, u)D_{u_j}.$$

(ii) Clear, since by (1.3.7b) $\mu(\tilde{a})$ intertwines the model and the quantum mapping cylinder of $\mu(T)$. \square

Remark: Statement (ii) is equivalent to

$$U_q(s+L, t^{-1}u) = e^{-ir_{kq}s} U_q(s, u)$$

(correcting the formula stated in [B.B., (9.3.25)].)

It follows that we may write a smooth function f in the adapted model in the form

$$(1.3.9) \quad f(s, u) = \sum_{k=0}^{\infty} \sum_{q \in \mathbb{N}^n} \hat{f}(k, q) e^{ir_{kq}s} U_q(s, u).$$

§1.4 Metric scaling and weights

As mentioned above, Δ and hence the wave invariants have well-defined weights under the metric rescaling $g \rightarrow \epsilon^2 g$. Since the wave invariants will be expressed in terms of QBNF coefficients, it is natural to ask how the latter scale. The question is not really well-posed since the QBNF coefficients are coefficients

with respect to Harmonic oscillators $\partial_{y_j}^2 + y_j^2$ whose scaling behaviour depends on the choice of coordinates. To amplify this point, we record how various metric objects scale under metric re-scaling.

In the following table, (s, y) denote the Fermi normal coordinates relative to g , (s, u) denote the scaled coordinates $(s, L^{-\frac{1}{2}}y)$ and p_u denotes the symplectic coordinates dual to u .

| g | $\epsilon^2 g$ |
|----------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------|
| $L, (s, y)$ | $\epsilon L, (\epsilon s, \epsilon y)$ |
| $\partial_s, \partial_{y_j}, e_j$ | $\epsilon^{-1}\partial_s, \epsilon^{-1}\partial_{y_j}, \epsilon^{-1}e_j$ |
| $y_{ij} := g(Y_i, e_j)$ | $\epsilon^{\frac{1}{2}}y_{ij} = \epsilon^2 g(\epsilon^{-\frac{1}{2}}Y_j, \epsilon^{-1}e_j)$ |
| $K_{ij} = g(R(\partial_s, e_i)\partial_s, e_j)$ | $\epsilon^{-2}K_{ij}$ |
| $u = L^{-\frac{1}{2}}y \quad p_u = L^{\frac{1}{2}}\eta$ | $\epsilon^{\frac{1}{2}}u \quad \epsilon^{-\frac{1}{2}}p_u$ |
| $\Lambda_j = \sum_{k=1}^n (iy_{jk}D_{u_k} - \dot{y}_{jk}u_k)$ | $\sum_{k=1}^n (iy_{jk}D_{u_k} - \dot{y}_{jk}u_k)$ |
| $\mathcal{L} := D_s - \frac{1}{2}(\sum_{j=1}^n D_{u_j}^2 + \sum_{i,j=1}^n K_{ij}(s)u_i u_j)$ | $\epsilon^{-1}[D_s - \frac{1}{2}(\sum_{j=1}^n D_{u_j}^2 + \sum_{i,j=1}^n K_{ij}(s)u_i u_j)]$ |

The entry $y_{ij} \rightarrow \epsilon^{\frac{1}{2}}y_{ij}$ follows from the scale invariance of the Jacobi equation together with the normalization condition

$$g(ReY_i, Im\dot{Y}_i) - g(Re\dot{Y}_i, ImY_i) = Constant.$$

which implies that $Y_i \rightarrow \epsilon^{-\frac{1}{2}}Y_i$.

We observe that the creation/annihilation operators, hence the harmonic oscillators $\Lambda_j^* \Lambda_j$, of the adapted model are scale-invariant, and that the distinguished element \mathcal{L} has weight -1. These are the desired scaling properties and we would like the basic and twisted models to possess them as well. As they stand, these models do not scale properly if we interpret the (s, y) -coordinates as Fermi normal coordinates. However, they do scale properly if we interpret the y coordinates as weightless.

§1.5 Scaled adapted model and intertwining operators

To avoid confusion, we now introduce the weightless coordinates $x = L^{-1}y = L^{-\frac{1}{2}}u, \xi = L\eta = L^{\frac{1}{2}}p_u$ and henceforth use them exclusively for the scaled adapted, basic and twisted models. In the following table we record how the various objects appear in the weightless coordinates. We also record the various intertwining operators, since they get altered when we used weightless coordinates. For instance, intertwining by $\mu(r_\alpha)$ above is weightless but that by $\mu(a)$ is of mixed weight.

| Adapted model | Scaled adapted model |
|---------------|-------------------------------------------------------------------------------------------------|
| Λ_j | $\sum_{k=1}^n (iy_{jk}L^{-\frac{1}{2}}D_{x_k} - L^{-\frac{1}{2}}\frac{dy_{jk}}{ds}x_k)$ |
| \mathcal{L} | $D_s - \frac{1}{2}(\sum_{j=1}^n L^{-1}D_{u_j}^2 + \sum_{i,j=1}^n LK_{ij}u_i u_j)$ |
| $U_o(s, u)$ | $U_{Lo}(s, x) = (detY(s))^{-1/2}e^{\frac{i}{2}\langle \Gamma(s)x, x \rangle} dx ^{\frac{1}{2}}$ |

The following intertwining operators will arise in the construction of the normal form:

| From * to * | Classical | Quantum |
|-------------------------|---------------------------------------------------------------------------------------------------------------------|------------------------------------|
| Ad.model to Sc.ad.model | $(u, p_u) \rightarrow (L^{-\frac{1}{2}}u, L^{\frac{1}{2}}p_u) = (x, \xi)$ | $\mu(D_{L^{\frac{1}{2}}})$ |
| Sc.Ad.Model to Tw.model | $(x, \xi) \rightarrow (L^{-\frac{1}{2}}ReYx + L^{-\frac{1}{2}}ImY\xi, L^{\frac{1}{2}}ReYx + L^{\frac{1}{2}}ImY\xi)$ | $\mu(D_{L^{\frac{1}{2}}} \cdot a)$ |

Above, the notation $\mu(D_r)$ refers to the metaplectic (dilation) operator corresponding to the symplectic matrix

$$D_r := \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \quad \mu(D_r)f(s, y) := f(s, r^{-1}y).$$

2. SEMI-CLASSICAL NORMAL FORM OF THE LAPLACIAN

We return now to $\sqrt{\Delta}$, which as in the introduction will be identified with its transfer to N_γ under the exponential map. The finite jets of this transfer are globally well-defined on N_γ , so we will often treat Δ as if it too were globally well-defined. Our purpose is to define the *semi-classically rescaled Laplacian* Δ_h and to put Δ_h into a *semi-classical normal form*. This is the crucial preliminary step in putting Δ itself into normal form.

In view of §1.5, there are two rescalings at hand: the semi-classical rescaling Δ_h and the metric L -rescaling above. The two rescalings have quite distinct origins, so we have kept them separate.

To motivate the rescalings and the emergence of semi-classical asymptotics, let us recall that the quasi-modes associated to γ have the form

$$(2.1) \quad \Phi_{kq}(s, \sqrt{r_{kq}}y) = e^{ir_{kq}s} \sum_{j=0}^{\infty} r_{kq}^{-\frac{j}{2}} U_q^{\frac{j}{2}}(s, \sqrt{r_{kq}}y, r_{kq}^{-1})$$

with $U_q^o = U_q$ (see [B.B]). The intertwining operator W_γ to the normal form is then the operator defined by the equations

$$(2.2) \quad W_\gamma \phi_{kq}(s, y) = \Phi_{kq}(s, \sqrt{r_{kq}}y).$$

The higher order terms $U_q^{\frac{j}{2}}$, hence W_γ , are determined by the conditions

$$(2.3) \quad \Delta_y e^{ir_{kq}s} U_{kq}(s, \sqrt{r_{kq}}y, r_{kq}^{-1}) \sim \lambda_{kq} e^{ir_{kq}s} U_{kq}(s, \sqrt{r_{kq}}y, r_{kq}^{-1})$$

with λ_{kq} given by (0.9).

Now write

$$r_{kq} = \frac{1}{h_{kq}L}, \quad h_{kq} := (2\pi k + \sum_{j=1}^n (q_j + \frac{1}{2}\alpha_j))^{-1}$$

so that the Planck constants h_{kq} are metric-independent. In the scaled Fermi coordinates u of §1.3-5, the quasi-modes have then the form

$$(2.1') \quad \Phi_{kq}(s, h^{-\frac{1}{2}}u) = e^{i\frac{s}{h_{kq}L}} U_{kq}(s, \sqrt{h_{kq}}^{-\frac{1}{2}}u, h_{kq})$$

and the eigenvalue problem (2.3) becomes

$$(2.3') \quad \Delta_u e^{i\frac{s}{h_{kq}L}} U(s, h_{kq}^{-\frac{1}{2}}u, h_{kq}) = \lambda(h_{kq}) e^{i\frac{s}{h_{kq}L}} U(s, h_{kq}^{-\frac{1}{2}}u, h_{kq}).$$

We are thus led to study the asymptotic eigenvalue problem

$$(2.4) \quad \Delta_u e^{i\frac{s}{hL}} U(s, h^{-\frac{1}{2}}u, h) = \lambda(h) e^{i\frac{s}{hL}} U(s, h^{-\frac{1}{2}}u, h)$$

on $C^\infty(\mathbb{R}^1 \times \mathbb{R}^n)$ with $U(s, u, h)$ and $\lambda(h)$ asymptotic series in h . Since Δ_u comes from an operator on N_γ , the eigenvalue problem is taking place on the quantized mapping cylinder \mathcal{H}_T (§1.3) of the adapted model.

We also note that the local expression for the Laplacian in the (s, u) coordinates is the same as in the (s, y) (Fermi) coordinates:

$$\Delta_u = \frac{1}{\sqrt{g}} \sum_{ij=0}^n \partial_{u_i} g^{ij} \sqrt{g} \partial_{u_j} = \Delta_y|_{u_j := L^{-\frac{1}{2}}y_j}$$

since $g^{ij} \rightarrow L^{-1}g^{ij}$ and $\partial_{u_i} \rightarrow L^{\frac{1}{2}}\partial_{y_i}$. Momentarily we are going to rescale the coordinates again to the weightless x -coordinates of §1.5, and again the Laplacian will be given by the usual expression. Hence it will be a simple matter to pass back and forth between the (s, y) , (s, u) and (s, x) expressions.

It is natural at this point to introduce the unitary operators T_h and M_h on \mathcal{H}_T or equivalently on the 1/2-density version $L_T^2(\mathbb{R}^1 \times \mathbb{R}^n, \Omega_{1/2})$ given by

$$(2.5a) \quad T_h(f(s, u)|ds|^{1/2}|du|^{1/2}) := h^{-n/2} f(s, h^{-\frac{1}{2}}u)|ds|^{1/2}|du|^{1/2}$$

$$(2.5b) \quad M_h(f(s, u)|ds|^{1/2}|du|^{1/2}) := e^{i\frac{s}{hL}} f(s, y)|ds|^{1/2}|du|^{1/2}$$

We easily see that:

$$(2.6) \quad T_h^* D_{u_j} T_h = h^{-\frac{1}{2}} D_{u_j}$$

$$T_h^* u_i T_h = h^{\frac{1}{2}} u_i$$

$$M_h^* D_s M_h = ((hL)^{-1} + D_s)$$

$$[M_h, u_i] = [M_h, D_{u_i}] = [M_h, T_h] = [T_h, D_s] = 0.$$

Definition The rescaling of an operator $A_u = a(s, D_s, u, D_u)$ of the adapted model is the operator

$$(2.7) \quad A_h := T_h^* M_h^* A T_h M_h$$

We observe that the operation of rescaling is weightless. In particular, the rescaled Laplacian Δ_h in the sense of (2.7) is of weight -2. To calculate it, we first note that the (1/2-density) Laplacian in scaled Fermi normal coordinates is given by the expression

$$(2.8) \quad \Delta_u = -(J^{-1} \partial_s J g^{oo} \partial_s + \sum_{ij=1}^n J^{-1} \partial_{u_i} g^{ij} J \partial_{u_j})$$

where $\partial_x := \frac{\partial}{\partial x}$, and where $J = J(s, u) = \sqrt{g}$ is the volume density in these coordinates. To obtain a self-adjoint operator with respect to the Lesbegue density $|ds| |du|$, we replace Δ by the unitarily equivalent 1/2-density Laplacian

$$\Delta_{1/2} := J^{1/2} \Delta J^{-1/2},$$

which can be written in the form:

$$(2.9) \quad \begin{aligned} -\Delta_{1/2} &= J^{-1/2} \partial_s g^{oo} J \partial_s J^{-1/2} + \sum_{ij=1}^n J^{-1/2} \partial_{u_i} g^{ij} J \partial_{u_j} J^{-1/2} \\ &\equiv g^{oo} \partial_s^2 + \Gamma^o \partial_s + \sum_{ij=1}^n g^{ij} \partial_{u_i} \partial_{u_j} + \sum_{i=1}^n \Gamma^i \partial_{u_i} + \sigma_o. \end{aligned}$$

From now on, we will only use $\Delta_{\frac{1}{2}}$ and denote it simply by Δ .

We then have:

$$(2.10) \quad -M_h^* \Delta M_h = -(hL)^{-2} g^{oo} + 2i(hL)^{-1} g^{oo} \partial_s + i(hL)^{-1} \Gamma^o + \Delta$$

Conjugation with T_h then gives

$$(2.11) \quad -\Delta_h = -(hL)^{-2} g_{[h]}^{oo} + 2i(hL)^{-1} g_{[h]}^{oo} \partial_s + i(hL)^{-1} \Gamma_{[h]}^o + h^{-1} \left(\sum_{ij=1}^n g_{[h]}^{ij} \partial_{u_i} \partial_{u_j} \right) + h^{-\frac{1}{2}} \left(\sum_{i=1}^n \Gamma_{[h]}^i \partial_{u_i} \right) + (\sigma)_{[h]},$$

the subscript $[h]$ indicating to dilate the coefficients of the operator in the form, $f_h(s, u) := f(s, h^{\frac{1}{2}} u)$.

Expanding the coefficients in Taylor series at $h = 0$, we obtain the asymptotic expansion

$$(2.12) \quad \Delta_h \sim \sum_{m=0}^{\infty} h^{(-2+m/2)} \mathcal{L}_{2-m/2}$$

where $\mathcal{L}_2 = L^{-2}$, $\mathcal{L}_{3/2} = 0$ and where

$$(2.13) \quad \mathcal{L}_1 = 2L^{-1} \left[i \frac{\partial}{\partial s} + \frac{1}{2} \left\{ \sum_{j=1}^n \partial_{u_j}^2 - \sum_{ij=1}^n K_{ij}(s) u_i u_j \right\} \right]$$

We observe that \mathcal{L}_1 is $2L^{-1}$ times the distinguished element \mathcal{L} of the adapted model, and that \mathcal{L} has weight -1 when the u -variables are given their natural weights $1/2$.

As discussed in §1. 4-5, it will be helpful to rescale the variables once again to make them weightless. Hence we change variables to $x = L^{-\frac{1}{2}}u$ and rewrite Δ_h and the $\mathcal{L}_{2-\frac{m}{2}}$'s in terms of the x -variables. For instance we will henceforth write \mathcal{L} in the form:

$$\mathcal{L} = i \frac{\partial}{\partial s} + \frac{1}{2} \left[\sum_{j=1}^n L^{-1} \partial_{x_j}^2 - \sum_{ij=1}^n L K_{ij}(s) x_i x_j \right].$$

We further note that the operators $\mathcal{L}_{2-m/2}$ now satisfy the periodicity condition (1.3.8ii) : Indeed, as noted above, Δ has this property when expressed in normal coordinates, and the various transversal rescalings and the conjugations by T_h, M_h preserve it. To indicate that an operator has this periodicity property and hence acts on \mathcal{H}_T , we will subscript the appropriate spaces of operators with a ‘‘T.’’

From (2.11) we see that the terms in $\mathcal{L}_{2-\frac{m}{2}}$ are of the form

$$\begin{array}{ccc} x^m & x^{m-2} & x^{m-4} \\ x^{m-2} D_x^2 & x^{m-3} D_x & \\ x^{m-2} D_s & x^{m-4} D_s & x^{m-4} D_s^2 \end{array}$$

hence

$$(2.14a) \quad \mathcal{L}_{2-\frac{m}{2}} \in \Psi^2(S^1)_T \otimes \mathcal{E}_\epsilon^{m-4} + \Psi^1(S^1)_T \otimes \mathcal{E}_\epsilon^{m-2} + \Psi^0(S^1)_T \otimes \mathcal{E}_\epsilon^m$$

where the subscript ϵ indicates that the Weyl symbol is a polynomial with the parity of m . Moreover, in the s variable, $\mathcal{L}_{2-\frac{m}{2}}$ is a polynomial in D_s of degree at most two, so we can refine (2.14a) to the statement

$$(2.14b) \quad \mathcal{L}_{2-\frac{m}{2}} \in C_T^\infty(S_L^1, \mathcal{E}_\epsilon^{m-4}) D_s^2 + C_T^\infty(S_L^1, \mathcal{E}_\epsilon^{m-2}) D_s + C_T^\infty(S_L^1, \mathcal{E}_\epsilon^m)$$

Comparing with (1.3.4) we see that

$$\mathcal{L} = \mu(\mathcal{A}_L^*) D_s \mu(\mathcal{A}_L^*)^{-1}$$

where \mathcal{A}_L is the weightless Wronskian matrix $D_{L^{\frac{1}{2}}} a^*$, that is

$$\mathcal{A}_L := \begin{pmatrix} L^{\frac{1}{2}} Im \dot{Y} & L^{\frac{1}{2}} Re \dot{Y} \\ L^{-\frac{1}{2}} Im Y & L^{-\frac{1}{2}} Re Y \end{pmatrix}.$$

This motivates the conjugation of (2.11) to the (untwisted) model. We therefore put

$$\mathcal{D}_h = \mu(\mathcal{A}_L^*)^{-1} \Delta_h \mu(\mathcal{A}_L^*)$$

which has the asymptotic expansion

$$(2.15) \quad \mathcal{D}_h \sim \sum_{m=0}^{\infty} h^{(-2+\frac{m}{2})} \mathcal{D}_{2-\frac{m}{2}}$$

with $\mathcal{D}_2 = I, \mathcal{D}_{\frac{3}{2}} = 0, \mathcal{D}_1 = D_s$. Conjugation by $\mu(\mathcal{A}_L^*)$ preserves weights, homogeneity and parity in the variables (x, D_x) . It also transforms D_s into D_s plus a term quadratic in (x, D_x) . Hence we find easily that $\mathcal{D}_{2-\frac{m}{2}}$ has weight -1 for each m and that

$$(2.16a) \quad \mathcal{D}_{2-\frac{m}{2}} \in \Psi^2(\mathbb{R}) \otimes \mathcal{E}_\epsilon^{m-4} + \Psi^1(\mathbb{R}) \otimes \mathcal{E}_\epsilon^{m-2} + \Psi^0(\mathbb{R}) \otimes \mathcal{E}_\epsilon^m$$

or, analogously to (2.14b),

$$(2.16b) \quad \mathcal{D}_{2-\frac{m}{2}} \in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m-4}) D_s^2 + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m-2}) D_s + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^m)$$

Of course, conjugation with $\mu(\mathcal{A}_L^*)$ also alters the periodicity property of the terms $\mathcal{D}_{2-\frac{m}{2}}$. From (1.3.7a) and (1.3.8) we see in fact that they transform like operators in the twisted model, i.e. on the quantum mapping cylinder of $r_\alpha(L)$. More precisely, we have

$$\begin{aligned} (2.17a) \quad \mathcal{D}_{2-\frac{m}{2}}|_{s+L} &= \mu(\mathcal{A}_L^*)^{-1} (\mathcal{L}_{2-\frac{m}{2}} \mu(\mathcal{A}_L^*)|_{(s+L)}) = \mu(r_\alpha(L)) \mu(\mathcal{A}_L^*)^{-1} \mu(T)^* \mathcal{L}_{2-\frac{m}{2}}|_{(s+L)} \mu(T) \mu(\mathcal{A}_L^*) \mu(r_\alpha(L))^* \\ &= \mu(r_\alpha(L)) \mu(\mathcal{A}_L^*)^* \mathcal{L}_{2-\frac{m}{2}} \mu(\mathcal{A}_L^*) \mu(r_\alpha(L))^* = \mu(r_\alpha(L)) \mathcal{D}_{2-\frac{m}{2}} \mu(r_\alpha(L))^*. \end{aligned}$$

Equivalently, in terms of the matrix elements in the basis of Hermite functions, we have

$$(2.17b) \quad \langle \mathcal{D}_j \gamma_q, \gamma_r \rangle(s+L) = e^{-i(\kappa_q - \kappa_r)s} \langle \mathcal{D}_j \gamma_q, \gamma_r \rangle(s)$$

To indicate that these operators act on \mathcal{H}_α we henceforth subscript the appropriate spaces of operators with an α .

To render these terms periodic in s , we have to conjugate further to the (untwisted) model under $\mu(r_\alpha)$. We record the resulting expressions, since they will be used later on. In the notation $\tilde{\mathcal{A}}_L^*(s) := \mathcal{A}_L^*(s) \cdot r_\alpha(s)$ the principal operator becomes, by Proposition (1.3.8),

$$(2.18) \quad \mu(\tilde{\mathcal{A}}_L^*) \mathcal{L} \mu(\tilde{\mathcal{A}})^{-1} = \mathcal{R}$$

Since conjugation by $\mu(r_\alpha)$ also preserves weight, homogeneity and parity, we further have:

$$(2.19) \quad \mathcal{R}_h := \mu(\tilde{a}) \Delta_h \mu(\tilde{a})^* \sim \sum_{m=0}^{\infty} h^{(-2+\frac{m}{2})} \mathcal{R}_{2-\frac{m}{2}}$$

with $\mathcal{R}_2 = I, \mathcal{R}_{\frac{3}{2}} = 0, \mathcal{R}_1 = \mathcal{R}$ and with all coefficients of weight -2 and periodic, that is,

$$(2.20a) \quad \mathcal{R}_{2-\frac{m}{2}} \in \Psi^2(S^1) \otimes \mathcal{E}_\epsilon^{m-4} + \Psi^1(S^1) \otimes \mathcal{E}_\epsilon^{m-2} + \Psi^0(S^1) \otimes \mathcal{E}_\epsilon^m$$

or, analogously to (2.14b),

$$(2.20b) \quad \mathcal{R}_{2-\frac{m}{2}} \in C^\infty(S_L^1, \mathcal{E}_\epsilon^{m-4}) \mathcal{R}^2 + C^\infty(S_L^1, \mathcal{E}_\epsilon^{m-2}) \mathcal{R} + C^\infty(S_L^1, \mathcal{E}_\epsilon^m)$$

Our aim is now to put Δ , or Δ_h for certain h , into a semi-classical normal form. This normal form will be, at first, only a formal normal form for Δ_h as a formal h -pseudo-differential operator. Here, a formal h -pseudodifferential operator of order m on \mathbb{R}^N is the Weyl quantization

$$(2.21a) \quad a^w(x, hD; h)u(x) = (2\pi h)^{-n} \int \int e^{\frac{i}{h}\langle x-y, \xi \rangle} a\left(\frac{1}{2}(x+y), \xi; h\right) u(y) dy d\xi$$

of an amplitude a belonging to the space $S_{cl}^m(T^*\mathbb{R}^N)$ of asymptotic sums

$$(2.21b) \quad a(x, \xi, h) \sim h^{-m} \sum_{j=0}^{\infty} a_j(x, \xi) h^j$$

with $a_j \in C^\infty(\mathbb{R}^{2N})$. Such operators form an algebra $\Psi_h^*(\mathbb{R}^N)$ under composition, with Ψ_h^m the subspace of m th order elements. (See [Sj] for further background and references). We will also be concerned with the slightly different situation of h -pseudodifferential operators on $S^1 \times \mathbb{R}^n$, which are defined similarly using local coordinates. Combining the h -filtration with the previous filtrations of $\Psi^*(\mathbb{R}) \otimes \mathcal{W}^*(\mathbb{R}^n)$ we get the triply filtered algebra

$$\begin{aligned} & \Psi_h^{(*,*,*)}(\mathbb{R} \times \mathbb{R}^n) \\ & \Psi_h^{(k,m,\frac{n}{2})} := h^k \Psi^m(\mathbb{R}) \otimes \mathcal{W}^{\frac{n}{2}}, \end{aligned}$$

with $k+m+\frac{n}{2}$ the total order of an element (and similarly with S^1 replacing \mathbb{R} .)

The following lemma will prepare for the normal form. We state it in terms of the \mathcal{R} -operators of the model since the periodicity properties are simplest there. The notation " $A|_o$ " will be used for the restriction of an operator $A \in \Psi^*(S^1 \times \mathbb{R}^n)$ of the model to elements of \mathcal{R} -weight zero. Equivalently, after conjugation by $\mu(r_\alpha)$ to the twisted model, to elements of D_s -weight zero, that is, to functions independent of s in \mathcal{H}_α . If we write the latter A in the form $A_2 D_s^2 + A_1 D_s + A_o$, then $A|_o$ is $A_o|_o$ (restricted to weight 0 elements). We have:

(2.22) **Lemma** *There exists an L -dependent h -pseudodifferential operator $W_h = W_h(s, x, D_x)$ on $L^2(S_L^1 \times \mathbb{R}^n)$ such that, for each $s \in S_L^1$,*

$$W_h(s, x, D_x) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$$

is unitary, and such that

$$W_h^* \mathcal{R}_h W_h \sim -h^{-2} L^{-2} + 2h^{-1} L^{-1} \mathcal{R} + \sum_{j=0}^{\infty} h^{\frac{j}{2}} \mathcal{R}_{2-\frac{j}{2}}^\infty(s, D_s, y, D_y)$$

where

$$(i) \quad \mathcal{R}_{2-\frac{j}{2}}^\infty(s, D_s, x, D_x) = \mathcal{R}_{2-\frac{j}{2}}^{\infty,2} \mathcal{R}^2 + \mathcal{R}_{2-\frac{j}{2}}^{\infty,1} \mathcal{R} + \mathcal{R}_{2-\frac{j}{2}}^{\infty,0}, \text{ with } \mathcal{R}_{2-\frac{j}{2}}^{\infty,k} \in C^\infty(S_L^1, \mathcal{E}_\epsilon^{j-2k});$$

(ii) $\mathcal{R}_{2-j}^\infty(s, D_s, x, D_x)|_o = \mathcal{R}_{2-j}^{\infty,o}(s, x, D_x)|_o = f_j(I_1, \dots, I_n)|_o$ for certain polynomials f_j of degree $j+2$ on \mathbb{R}^n , i.e. $f_j(I_1, \dots, I_n) \in \mathcal{P}_T^{j+2}$

(iii) $\mathcal{R}_{2-\frac{2k+1}{2}}^\infty(s, D_s, x, D_x)|_o = \mathcal{R}_{2-\frac{2k+1}{2}}^{\infty,o}(s, x, D_x)|_o = 0$;

(iv) The harmonic oscillators are weightless and all of the \mathcal{R} 's have weight -2 .

Proof: The operator W_h will be constructed as the asymptotic product

$$(2.23) \quad W_h := \mu(r_\alpha)^* \Pi_{k=1}^\infty W_{h^{\frac{k}{2}}} \mu(r_\alpha)$$

of weightless unitary h -pseudodifferential operators on \mathbb{R}^n , with

$$(2.23a) \quad W_{h^{\frac{k}{2}}} := \exp(ih^{\frac{k}{2}} Q_{\frac{k}{2}})$$

and with $h^{\frac{k}{2}} Q_{\frac{k}{2}} \in h^{\frac{k}{2}} \mathcal{C}^\infty(S_L^1) \otimes \mathcal{E}^{k+2}$ of total order 1. The product will converge, for each s , to a unitary operator in $\Psi_h^o(\mathbb{R}^n)$ (see [Sj] for discussion of asymptotic products).

To see what is involved, we first construct a weightless $Q_{\frac{1}{2}}(s, x, D_x) \in C^\infty(S_L^1) \otimes \mathcal{E}_\epsilon^3$ such that

$$(2.24a) \quad e^{-ih^{\frac{1}{2}} Q_{\frac{1}{2}}} \mathcal{R}_h e^{ih^{\frac{1}{2}} Q_{\frac{1}{2}}}|_o = [-h^{-2} L^{-2} + 2h^{-1} L^{-1} \mathcal{R} + \mathcal{R}_o^{\frac{1}{2}} + \dots]|_o$$

where the dots \dots indicate higher powers in h . The operator $Q_{\frac{1}{2}}$ then must satisfy the commutation relation

$$(2.24b) \quad \{[L^{-1} \mathcal{R}, Q_{\frac{1}{2}}] + \mathcal{R}_{\frac{1}{2}}\}|_o = 0.$$

To solve for $Q_{\frac{1}{2}}$, it is convenient to conjugate back to the $\mathcal{D}_{2-\frac{m}{2}}$'s of the twisted model by $\mu(r_\alpha)$, since this transforms \mathcal{R} into D_s . The commutation relation thus becomes

$$(2.24c) \quad \{[L^{-1} D_s, \mu(r_\alpha)^* Q_{\frac{1}{2}} \mu(r_\alpha)] + \mathcal{D}_{\frac{1}{2}}\}|_o = 0,$$

that is,

$$(2.24d) \quad L^{-1} \partial_s \{\mu(r_\alpha)^* Q_{\frac{1}{2}} \mu(r_\alpha)\}|_o = -i \{\mathcal{D}_{\frac{1}{2}}\}|_o$$

where $\partial_s A$ is the Weyl operator whose complete symbol is the s -derivative of that of A . Since (2.24d) is simpler than (2.24b), we henceforth conjugate everything by $\mu(r_\alpha)$, and relabel the operators $\mu(r_\alpha)^* Q \mu(r_\alpha)$ by \tilde{Q} . The resulting \mathcal{D} 's then have the twisted model periodicity properties (2.17a-b). Our problem is then to solve (2.24d) with an operator $\tilde{Q}_{\frac{1}{2}}$ satisfying (2.17a-b).

To do so, we first observe that under conjugation by $\mu(r_\alpha)$, elements of \mathcal{R} -weight zero transform to elements of D_s -weight zero, and hence it suffices to solve for the matrix elements $\langle \tilde{Q}_{\frac{1}{2}} \gamma_q, \gamma_r \rangle$. It follows from (2.17b) that the solution is unique for $r \neq q$, while for $r = q$ the matrix element $\langle \tilde{Q}_{\frac{1}{2}} \gamma_q, \gamma_q \rangle$ is periodic and hence a necessary and sufficient condition for solvability by an operator with the correct periodicity is that, for all $q \in \mathbb{N}^n$,

$$(2.25). \quad \int_o^L \langle \mathcal{D}_{\frac{1}{2}} \gamma_q, \gamma_q \rangle ds = 0$$

In view of (2.16a-b), we have

$$\mathcal{D}_{\frac{1}{2}} \in \Psi^1(\mathbb{R})_\alpha \otimes \mathcal{E}_\epsilon^1 + \Psi^o(\mathbb{R})_\alpha \otimes \mathcal{E}_\epsilon^3,$$

in fact a simple calculation shows that $\mathcal{D}_{\frac{1}{2}} \in \Psi^o(\mathbb{R})_\alpha \otimes \mathcal{E}_\epsilon^3$. (Recall that the subscript α indicates the periodicity property (2.17a-b).)

We now observe that if $A \in \Psi^*(\mathbb{R}) \otimes \mathcal{E}_\epsilon^m$ then

$$(2.26) \quad A : \left\{ \begin{array}{ll} L_\pm^2 \rightarrow L_\pm^2 & m \text{ even} \\ : L_\pm^2 \rightarrow L_\mp^2 & m \text{ odd} \end{array} \right\},$$

where L_\pm^2 (resp. L_-^2) denotes the subspace of $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n)$ spanned by even (resp. odd) functions of $y \in \mathbb{R}^n$ with coefficients in s .

It follows that

$$\mathcal{D}_{\frac{1}{2}} : L_+^2 \rightarrow L_-^2,$$

This parity reversing property implies the vanishing of the diagonal matrix elements in the basis $\{\gamma_q\}$, hence (2.25) does hold. A unique solution of (2.24d) is specified by the condition that

$$\int_o^{2\pi} \langle \tilde{Q}_{\frac{1}{2}} \gamma_q, \gamma_q \rangle ds = 0.$$

To solve the equation (2.24d) we rewrite it in terms of complete Weyl symbols. We will use the notation $A(s, x, \xi)$ for the complete Weyl symbol of the operator $A(s, x, D_x)$. Then (2.24d) becomes

$$(2.27a) \quad L^{-1} \partial_s \tilde{Q}_{\frac{1}{2}}(s, x, \xi) = -i \mathcal{D}_{\frac{1}{2}}|_o(s, x, \xi)$$

with

$$\tilde{Q}_{\frac{1}{2}}(s + L, x, \xi) = \tilde{Q}_{\frac{1}{2}}(s, r_\alpha(L)(x, \xi)).$$

We solve (2.27a) with the Weyl symbol

$$\tilde{Q}_{\frac{1}{2}}(s, x, \xi) = \tilde{Q}_{\frac{1}{2}}(0, x, \xi) + L \int_0^s -i \mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi) du$$

where $\tilde{Q}_{\frac{1}{2}}(0, x, \xi)$ is determined by the consistency condition

$$(2.27b) \quad \tilde{Q}_{\frac{1}{2}}(L, x, \xi) - \tilde{Q}_{\frac{1}{2}}(0, x, \xi) = L \int_0^L -i \mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi) du$$

or in view of the periodicity condition in (2.27a),

$$(2.27c) \quad \tilde{Q}_{\frac{1}{2}}(0, r_\alpha(x, \xi)) - \tilde{Q}_{\frac{1}{2}}(0, x, \xi) = L \int_0^L -i \mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi) du.$$

To solve, we use that $\mathcal{D}_{\frac{1}{2}}|_o(u, x, \xi)$ is a polynomial of degree 3 in (x, ξ) . Also, as is customary in such calculations, we switch to complex coordinates $z_j = x_j + i\xi_j$ and $\bar{z}_j = x_j - i\xi_j$ in which the action of $r_\alpha(L)$ is diagonal. With a little abuse of notation, we will continue to denote the Weyl symbols, qua functions of the z_j, \bar{z}_j 's by their previous expressions. We also suppress the subscripts by using vector notation z, \bar{z} and $e^{i\alpha}$. Thus, (2.27c) becomes

$$(2.28) \quad \tilde{Q}_{\frac{1}{2}}(0, e^{i\alpha} z, e^{-i\alpha} \bar{z}) - \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z}) = L \int_0^L -i \mathcal{D}_{\frac{1}{2}}|_o(u, z, \bar{z}) du.$$

We now use that $\mathcal{D}_{\frac{1}{2}}(u, z, \bar{z})$ is a polynomial of degree 3 to solve (2.27c). If we put

$$\tilde{Q}_{\frac{1}{2}}(s, z, \bar{z}) = \sum_{|m|+|n|\leq 3} q_{\frac{1}{2};mn}(s) z^m \bar{z}^n$$

and

$$\mathcal{D}_{\frac{1}{2}}|_o(s, z, \bar{z}) du = \sum_{|m|+|n|\leq 3} d_{\frac{1}{2};mn}(s) z^m \bar{z}^n$$

then (2.28) becomes

$$(2.29). \quad \sum_{|m|+|n|\leq 3} (1 - e^{(m-n)\alpha}) q_{\frac{1}{2};mn}(0) z^m \bar{z}^n = -iL^2 \sum_{|m|+|n|\leq 3} \bar{d}_{\frac{1}{2};mn} z^m \bar{z}^n$$

Since there are no terms with $m = n$ in this (odd-index) equation, and since the α_j 's are independent of π over \mathbb{Z} , there is no obstruction to the solution of (2.29).

For simplicity of notation we will express the solution in the form

$$(2.30) \quad \tilde{Q}_{\frac{1}{2}}|_o = -iL \int \mathcal{D}_{\frac{1}{2}}(s) ds|_o$$

where \int denotes the integration procedure just defined, that is, the indefinite integral satisfying (2.27b). Since the integration only involves the s -coefficients, the solution is a pseudodifferential operator on \mathbb{R}^n with the same order, same order of vanishing, and same parity as the restriction of $\mathcal{D}_{\frac{1}{2}}$ to elements of weight zero. We then extend it all of to \mathcal{H}_α as a pseudodifferential operator with the same properties by stipulating that

$$(2.31a) \quad \tilde{Q}_{\frac{1}{2}} M_h = M_h \tilde{Q}_{\frac{1}{2}}.$$

Then, as desired,

$$(2.31b) \quad \tilde{Q}_{\frac{1}{2}} \in \Psi^o(\mathbb{R}^1) \otimes \mathcal{E}_\epsilon^3.$$

The conjugate by $\mu(r_\alpha)$ then defines a periodic operator satisfying (2.24b) and hence a unitary element $W_{h\frac{1}{2}} \in \Psi_h^o(S^1 \times \mathbb{R}^n)$ satisfying (2.24a). The twisted unitary operator with exponent $\tilde{Q}_{\frac{1}{2}}$, i.e. the image of $W_{h\frac{1}{2}}$ under conjugation by $\mu(r_\alpha)$, will be decorated with a tilde, $\tilde{W}_{h\frac{1}{2}}$.

Since $h^{\frac{1}{2}}\tilde{Q}_{\frac{1}{2}}$ is of total order 1, $h^{\frac{1}{2}}ad(\tilde{Q}_{\frac{1}{2}})$ (with $ad(A)B := [B, A]$) preserves the total order in $\Psi_h^{(*,*,*)}$, and hence $\tilde{W}_{h\frac{1}{2}}$ is an order-preserving automorphism. It is moreover independent of D_s and has an odd polynomial Weyl symbol, so that

$$(2.32) \quad h^{\frac{1}{2}}ad(\tilde{Q}_{\frac{1}{2}}) : h^{\frac{k}{2}}\Psi^l(\mathbb{R}) \otimes \mathcal{E}_\epsilon^m \rightarrow h^{\frac{k+1}{2}}[\Psi^{l-1}(\mathbb{R}) \otimes \mathcal{E}_\epsilon^{m+3} + \Psi^l(\mathbb{R}) \otimes \mathcal{E}_\epsilon^{m+1}].$$

Finally, since $\mathcal{D}_{\frac{1}{2}}$ has weight -2, L has degree 1 and since $\int_o^s(\cdot)ds$ has degree 1 in the metric scaling, we see that $\tilde{Q}_{\frac{1}{2}}$ is weightless. Alternatively, the $d_{\frac{1}{2};m,n}$'s have weight -2, the variables z have weight 0 and hence the $q_{\frac{1}{2};m,n}$'s have weight 0.

Consider now the element

$$\mathcal{D}_h^{\frac{1}{2}} := \tilde{W}_{h\frac{1}{2}}^* \mathcal{D}_h \tilde{W}_{h\frac{1}{2}} \in \Psi_h^2(\mathbb{R}^1 \times \mathbb{R}^n).$$

Using only the h -filtration, we expand

$$(2.33) \quad \begin{aligned} \mathcal{D}_h^{\frac{1}{2}} &\sim \sum_{n=0}^{\infty} h^{-2+\frac{n}{2}} \sum_{j+m=n} \frac{i^j}{j!} (ad\tilde{Q}_{\frac{1}{2}})^j \mathcal{D}_{2-\frac{m}{2}} \\ &:= h^{-2}L^{-2} + h^{-1}L^{-1}D_s + \sum_{n=3}^{\infty} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{1}{2}}. \end{aligned}$$

An obvious induction using (2.32) and (2.16a-b) gives that

$$ad(\tilde{Q}_{\frac{1}{2}})^j \mathcal{D}_{2-\frac{m}{2}} \in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m+j-4})D_s^2 + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m+j-2})D_s + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m+j}).$$

It follows that $\mathcal{D}_{2-\frac{n}{2}}^{\frac{1}{2}}$ has the same filtered structure (2.16b) as $\mathcal{D}_{2-\frac{n}{2}}$.

We carry this procedure out one more step before arguing inductively, since the even steps behave differently from the odd ones. We thus seek an element $\tilde{Q}_1(s, x, D_x) \in \Psi^*(S^1 \times \mathbb{R}^n)$ and an element $f_o(I_1, \dots, I_n) \in \mathcal{A}$ so that

$$\mathcal{D}_h^1 := \tilde{W}_{h1}^* \mathcal{D}_h^{\frac{1}{2}} \tilde{W}_{h1} = h^{-2}L^{-2} + h^{-1}L^{-1}D_s + h^{-\frac{1}{2}}\mathcal{D}_{\frac{1}{2}}^{\frac{1}{2}} + \mathcal{D}_o^1(s, D_s, x, D_x) + \dots$$

with

$$(2.34a) \quad \mathcal{D}_o^1(s, D_s, x, D_x)|_o = f_o(I_1, \dots, I_n)$$

with $\tilde{W}_{h1} = e^{ih\tilde{Q}_1}$, and where the dots signify terms of higher order in h . Note that $\mathcal{D}_{\frac{1}{2}}^1 = \mathcal{D}_{\frac{1}{2}}^{\frac{1}{2}}$, so that (2.33a) would imply

$$(2.34b) \quad \{h^{\frac{1}{2}}\mathcal{D}_{\frac{1}{2}}^1 + \mathcal{D}_o^1\}|_o = f_o(I_1, \dots, I_n).$$

The condition on \tilde{Q}_1 is then

$$(2.35a) \quad \{[D_s, \tilde{Q}_1] + \mathcal{D}_o^{\frac{1}{2}}\}|_o = f_o(I_1, \dots, I_n)$$

or equivalently

$$(2.35b) \quad \partial_s \tilde{Q}_1|_o = \{-\mathcal{D}_o^{\frac{1}{2}} + f_o(I_1, \dots, I_n)\}|_o$$

As above, we first consider the solvability of (2.35b) one matrix element at a time. The off-diagonal matrix element equations again have a unique solution, but because $\mathcal{D}_o^{\frac{1}{2}}$ is even there is now a condition on the solvability, with a periodic solution, of the diagonal ones:

$$(2.36) \quad \frac{1}{L} \int_o^L \langle \mathcal{D}_o^{\frac{1}{2}} \gamma_q, \gamma_q \rangle ds = f_o(q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}).$$

These conditions (2.36) determine f_o and hence the operator $f_o(I_1, \dots, I_n)$. To analyse the properties of f_o and of \tilde{Q}_1 , we use (1.1b), (2.16b) and (2.31c) to conclude that

$$\begin{aligned} \mathcal{D}_o^{\frac{1}{2}} &= \mathcal{D}_o + ad\tilde{Q}_{\frac{1}{2}}(ad\tilde{Q}_{\frac{1}{2}}D_s + \mathcal{D}_{\frac{1}{2}}) \\ &\in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^o)D_s^2 + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^2)D_s + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^4). \end{aligned}$$

We also observe that (2.32) is equivalent to

$$(2.37) \quad f_o(I_1, \dots, I_n) = \frac{1}{L} \int_o^L \int_{T^n} V_t^* \mathcal{D}_o^{\frac{1}{2}}|_o V_t dt ds$$

where $T^n = \mathbb{R}^n / \mathbb{Z}^n$ is the n -torus, where $t \in T^n$ and where

$$V(t_1, \dots, t_n) := \exp it_1 I_1 \cdots \exp it_n I_n.$$

Since V_t belongs to the metaplectic representation, we have by metaplectic covariance of the Weyl calculus that

$$(2.38) \quad f_o(I_1, \dots, I_n) \in \mathcal{P}_T^2 \subset \mathcal{E}^4$$

We then can solve for \tilde{Q}_1 in the form

$$(2.39) \quad \tilde{Q}_1(s, y, D_y) := -iL \left[\int \{ \mathcal{D}_o^{\frac{1}{2}} - f_o(I_1, \dots, I_n) \} |_o ds \right] \in \Psi^o(\mathbb{R}^1) \otimes \mathcal{E}_\epsilon^4$$

where \int is the indefinite integration in s with zero mean value and where the operator on the right is extended to $L^2(\mathbb{R}^1 \times \mathbb{R}^n)$ using (2.35b).

To make the integration more concrete and to analyze the solution, we again express everything in terms of complete Weyl symbols. Thus we rewrite (2.35b) in the form

$$(2.40a) \quad L^{-1} \partial_s \tilde{Q}_1(s, z, \bar{z}) = -i \{ \mathcal{D}_o^{\frac{1}{2}}|_o(s, z, \bar{z}) - f_o(|z_1|^2, \dots, |z_n|^2) \}$$

or equivalently

$$(2.40b) \quad \tilde{Q}_1(s, z, \bar{z}) = \tilde{Q}_1(0, z, \bar{z}) - iL \int_0^s [\mathcal{D}_o^{\frac{1}{2}}|_o(u, z, \bar{z}) - f_o(|z_1|^2, \dots, |z_n|^2)] du$$

and solve simultaneously for \tilde{Q}_1 and f_o . The consistency condition determining a unique solution is that

$$(2.41a) \quad \tilde{Q}_1(L, z, \bar{z}) = \tilde{Q}_1(0, z, \bar{z}) - iL \int_0^L [\mathcal{D}_o^{\frac{1}{2}}|_o(u, z, \bar{z}) - f_o(|z_1|^2, \dots, |z_n|^2)] du.$$

or in view of the twisted periodicity condition

$$(2.41b) \quad \tilde{Q}_1(0, e^{i\alpha} z, e^{-i\alpha} \bar{z}) - \tilde{Q}_1(0, z, \bar{z}) = -iL \left\{ \int_0^L \mathcal{D}_o^{\frac{1}{2}}|_o(u, z, \bar{z}) du - L f_o(|z_1|^2, \dots, |z_n|^2) \right\}.$$

As before we use that $\mathcal{D}_o^{\frac{1}{2}}|_o(u, z, \bar{z})$ is a polynomial of degree 4 to solve the equation. We put

$$(2.42) \quad \tilde{Q}_1(s, z, \bar{z}) = \sum_{|m|+|n| \leq 4} q_{1;mn}(s) z^m \bar{z}^n, \quad f_o(|z_1|^2, \dots, |z_n|^2) = \sum_{|k| \leq 2} c_{ok} |z|^{2k}$$

and

$$\mathcal{D}_o^{\frac{1}{2}}|_o(s, z, \bar{z}) du := \sum_{|m|+|n| \leq 4} d_{o;mn}^{\frac{1}{2}}(s) z^m \bar{z}^n, \quad \bar{d}_{o;mn}^{\frac{1}{2}} := \frac{1}{L} \int_o^L d_{o;mn}^{\frac{1}{2}}(s) ds$$

As above, we can solve for the off-diagonal coefficients,

$$(2.43a) \quad q_{1;mn}(0) = -iL^2 (1 - e^{i(m-n)\alpha})^{-1} \bar{d}_{o;mn}^{\frac{1}{2}}$$

and must set the diagonal coefficients equal to zero. The coefficients c_{ok} are then determined by

$$(2.43b) \quad c_{ok} = \bar{d}_{1;kk}^{\frac{1}{2}}.$$

It is evident that \tilde{Q}_1 and $f_o(I_1, \dots, I_n)$ are even polynomial pseudodifferential operators of degree 4 in the variables (x, D_x) , that \tilde{Q}_1 is weightless under metric rescalings and that the coefficients c_{ok} are of weight -2. They are essentially the same as the residual QBNF invariants.

We now proceed inductively to define self-adjoint polynomial pseudodifferential operators $\tilde{Q}_{\frac{j}{2}}, f_j(I_1, \dots, I_n)$, unitary h -pseudodifferential operators $\tilde{W}_{h^{\frac{j}{2}}} := \exp(ih^{\frac{j}{2}} \tilde{Q}_{\frac{j}{2}})$ and approximate semi-classical normal forms $\mathcal{D}_h^{\frac{k}{2}}$ such that:

(2.44)

- (i) $\mathcal{D}_h^{\frac{k}{2}} := \tilde{W}_{h^{\frac{k}{2}}}^* \tilde{W}_{h^{\frac{k-1}{2}}}^* \dots \tilde{W}_{h^{\frac{1}{2}}}^* \mathcal{D}_h \tilde{W}_{h^{\frac{1}{2}}} \dots \tilde{W}_{h^{\frac{k-1}{2}}} \tilde{W}_{h^{\frac{k}{2}}}$;
- (ii) $\mathcal{D}_h^{\frac{k}{2}} \sim h^{-2} + h^{-1} D_s + \sum_{n=3}^{\infty} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{k}{2}}$;
- (iii) for $k \geq n-2$, $\mathcal{D}_{2-\frac{n}{2}}^{\frac{k}{2}} = \mathcal{D}_{2-\frac{n}{2}}^{\frac{n-2}{2}} = [D_s, \tilde{Q}_{\frac{n-1}{2}}] + \mathcal{D}_{-\frac{n-3}{2}}^{\frac{n-3}{2}}$;
- (iv) for $k \geq n-2$, $\mathcal{D}_{2-\frac{n}{2}}^{\frac{k}{2}}|_o \begin{cases} = f_j(I_1, \dots, I_n) & n = 2j \\ = 0 & n = 2j+1 \end{cases}$
- (v) $\mathcal{D}_{2-\frac{m}{2}}^{\frac{k}{2}} \in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m-4}) D_s^2 + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{m-2}) D_s + C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^m)$;
- (vi) $\tilde{Q}_{\frac{k}{2}} \in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{k+2})$;
- (vii) the conjugates under $\mu(r_\alpha)$ of the above operators are periodic of period L in s ;
- (viii) $f_j(I_1, \dots, I_n) \in \mathcal{P}_T^{j+2}$; (ix) $\text{wgt}(f_j) = -2$.

To check the induction, let us assume these properties hold for $k \leq N$. Then

$$(2.45) \quad \begin{aligned} \mathcal{D}_h^{\frac{N+1}{2}} &= \tilde{W}_{h^{\frac{N+1}{2}}}^* \mathcal{D}_h^{\frac{N}{2}} \tilde{W}_{h^{\frac{N+1}{2}}} = \\ &= h^{-2} L^{-2} + h^{-1} L^{-1} D_s + \sum_{n=3}^{N+2} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{N}{2}} + h^{\frac{N-1}{2}} \{[L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}\} + \sum_{n=N+4}^{\infty} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{N}{2}} \\ &+ \sum_{m=2}^{\infty} h^{m-\frac{N+1}{2}} \frac{i^m}{m!} (ad \tilde{Q}_{\frac{N+1}{2}})^m (D_s) + \sum_{m=1}^{\infty} \sum_{p=3}^{\infty} h^{m-\frac{N+1}{2}+\frac{p}{2}-2} \frac{i^m}{m!} (ad \tilde{Q}_{\frac{N+1}{2}})^m (\mathcal{D}_{2-\frac{p}{2}}^{\frac{N}{2}}). \end{aligned}$$

We note that

$$\mathcal{D}_{2-\frac{N+3}{2}}^{\frac{N+1}{2}} = [L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}$$

and that further conjugations will not alter this (or lower order) term(s), proving (iii).

When $N+1$ is odd, $\tilde{Q}_{\frac{N+1}{2}}$ must solve:

$$(2.46a) \quad \{[L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}\}|_o = 0$$

while if $N+1$ is even $\tilde{Q}_{\frac{N+1}{2}}$ and $f_{\frac{N-1}{2}}$ must solve

$$(2.46b) \quad \{[L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}\}|_o = f_{\frac{N-1}{2}}(I_1, \dots, I_n).$$

If $N+1$ is odd, (2.45v) implies that the diagonal matrix elements of $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}|_o$ vanish. Hence, as in the case of $\tilde{Q}_{\frac{1}{2}}$, the solution of (2.46a) is given by

$$(2.47a) \quad \tilde{Q}_{\frac{N+1}{2}}|_o := -iL \int \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}|_o ds$$

where the constant of integration $\tilde{Q}_{\frac{N+1}{2}}|_o(0, z, \bar{z})$ is defined so that the solution satisfies the periodicity condition analogous to (2.28). In the even case, as in the case of \tilde{Q}_1 we set

$$(2.47b) \quad f_{\frac{N-1}{2}}(q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}) := \frac{1}{L} \int_o^L \langle \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} \gamma_q, \gamma_q \rangle ds,$$

or equivalently

$$(2.47c) \quad f_{\frac{N-1}{2}}(I_1, \dots, I_n) := \frac{1}{L} \int_o^L \int_{T^n} V_t^* \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}|_o V_t dt ds,$$

and

$$(2.47d) \quad \tilde{Q}_{\frac{N+1}{2}}|_o := -L \int \{ \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} - f_{\frac{N-1}{2}}(I_1, \dots, I_n) \}|_o ds.$$

As above, $\tilde{Q}_{\frac{N+1}{2}}|_o$ is then extended to all of \mathcal{H}_α thru (2.31b). The indefinite integrations can be precisely defined, as above, by expressing everything as a polynomial in z, \bar{z} and solving the resulting algebraic equations for the constant of integration. By (2.47 c-d) the solution has the parity of $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}$, i.e. the parity of $N-1$, and by (2.45) the parity property propagates to the case $N+1$. The formula (2.47c) also shows that for even $N-1$, $f_{\frac{N-1}{2}}(I_1, \dots, I_n)$ has the same order and weight properties as $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}}$. Since the latter may be written in the form $aD_s^2 + bD_s + c$, with $c \in C^\infty(\mathbb{R}, \mathcal{E}_\epsilon^{N+3})$, the former lies in $\mathcal{P}_\mathcal{I}^* \cap \mathcal{E}_\epsilon^{N+3}$, implying (viii). The formula (2.47d) then implies (vi). The periodicity property (vii) is maintained throughout.

It follows immediately that

$$(2.48) \quad \begin{aligned} \mathcal{D}_{2-\frac{n}{2}}^\infty &= \mathcal{D}_{2-\frac{n}{2}}^{\frac{n-2}{2}} \\ &= [D_s, \tilde{Q}_{\frac{n}{2}-1}] + \mathcal{D}_{2-\frac{n}{2}}^{\frac{n-3}{2}}. \end{aligned}$$

Conjugating back under $\mu(r_\alpha)$, we see that statement (i) of the Lemma then follows from (2.48) and from (2.44iii-iv) and statement (ii) then follows from (2.44 viii). The weight property (ix) is visible from (2.47c). \square

As a transitional step to the quantum Birkhoff normal form, let us rephrase the above Lemma in terms of the actual Fermi normal coordinates and inverse Planck constants $\{r_{kq}\}$ in place of $(Lh)^{-1}$. We thus introduce the space $\mathcal{O}^*(N_\gamma, \Gamma, \{r_{kq}\})$ of semi-classical Hermite distributions associated to the non-homogenous isotropic manifold $\Gamma := \mathbb{R}^{+\gamma}$, i.e.

$$\Gamma := \{(s, \sigma, y, \eta) \in T^*(N_\gamma) : \sigma = 1, (y, \eta) = (0, 0)\}.$$

It is, by definition, the union of the spaces $\mathcal{O}^{m/2}$ of elements of order $\frac{m}{2}, m \in \mathbb{N}$, given in Fermi normal coordinates by asymptotic sums of the form

$$u(s, y, r_{kq}^{-1}) \sim (r_{kq})^{\frac{m}{2}} e^{ir_{kq}s} \sum_{n=0}^{\infty} (r_{kq})^{-\frac{n}{2}} f_n(s, \sqrt{r_{kq}}y)$$

with $f_n \in C_q^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$. Here, $\mathcal{S}(\mathbb{R}^n)$ is the Schwarz space, and the subscript q denotes the space of functions of this form satisfying

$$f(s + L, y) = e^{i\kappa_q s} f(s, ty).$$

Aside from the restriction to Schwartz functions and the half-integral orders, it is the image under the rescaling operator $T_{(r_{kq})^{-1}}$ of the space $\mathcal{O}^*(N_\gamma, \Lambda, \{r_{kq}\})$ of semi-classical Lagrangean distributions associated to the non-homogenous Lagrangean

$$\Lambda := \text{graph}(ds) = \{(s, \sigma, y, \eta) : \sigma = 1, \eta = 0\}.$$

Similar spaces of oscillatory functions could be, and implicitly are, defined in the model $S^1 \times \mathbb{R}^n$ but the scaling aspect comes out most naturally on N_γ . (For background on semi-classical Lagrangean distributions, see [CV2]).

Let us denote by $\mathcal{O}_o^*(N_\gamma, \Lambda, \{r_{kq}\})$ the subspace of elements of $\mathcal{O}^*(S^1 \times \mathbb{R}^n, \Lambda)$ satisfying $\mathcal{L}u(s, y, r_{kq}^{-1}) \sim 0$. Let us also denote by W_{kq} the transfer of $W_{r_{kq}^{-1}}$ above to the normal bundle, i.e.

$$W_{kq} := \mu(\tilde{a}_s) W_{r_{kq}^{-1}} \mu(\tilde{a}_s)^*.$$

For the sake of simplicity we will be a little negligent here of the rescalings. Under this operator, the kernel of \mathcal{L} goes over to the space $\tilde{\mathcal{O}}_o^*(N_\gamma, \Lambda, \{r_{kq}\})$ of elements annihilated by $W_{kq}^* \mathcal{L} W_{kq}$. It is clear from the above lemma that this space is stable under $W_{kq} * \Delta_{r_{kq}^{-1}} W_{kq}$. We may interpret what was proved in the lemma as giving a semi-classical normal form for Δ or $\Delta_{r_{kq}^{-1}}$ in the following sense:

(2.49) **Theorem** *The r_{kq}^{-1} -pseudodifferential operator*

$$W_{kq} : \mathcal{O}_o^*(N_\gamma, \Lambda, \{r_{kq}\}) \rightarrow \tilde{\mathcal{O}}_o^*(N_\gamma, \Lambda, \{r_{kq}\})$$

has the properties:

$$\begin{aligned}
W_{kq}^* W_{kq}^{-1} - I &\sim W_{kq} W_{kq}^* - I \sim 0 \\
W_{kq}^* T_{r_{kq}}^{-1} \Delta T_{r_{kq}}^{-1} W_{kq} &\sim \mathcal{L}^2 + f_o(I_{\gamma 1}, \dots, I_{\gamma n}) + \frac{f_1(I_{\gamma 1}, \dots, I_{\gamma n})}{\mathcal{L}} + \dots \\
W_{kq} e^{ir_{kq}s} U_q(s, y) &= e^{ir_{kq}s} W_{kq} U_q(s, y) \Phi_{kq}(s, y)
\end{aligned}$$

(see (2.1)).

3. NORMAL FORM: PROOF OF THEOREM B

The intertwining operators W_{kq} will now be assembled into the Fourier-Hermite -series integral operator

$$\begin{aligned}
(3.1) \quad W_\gamma : L^2(S_L^1 \times \mathbb{R}^n, ds dy) &\rightarrow L^2(S_L^1 \times \mathbb{R}^n, ds dy) \\
W_\gamma \sum_{(k,q) \in \mathbb{N}^{n+1}} \hat{f}(k, q) e^{ir_{kq}s} U_q(s, y) &= \sum_{(k,q) \in \mathbb{N}^{n+1}} \hat{f}(k, q) e^{ir_{kq}s} W_{kq} U_q(s, y).
\end{aligned}$$

Also, the dilation operators will be assembled into the operator

$$\begin{aligned}
(3.2) \quad T : L^2(S_L^1 \times \mathbb{R}^n, ds dy) &\rightarrow L^2(S_L^1 \times \mathbb{R}^n, ds dy) \\
T \sum_{(k,q) \in \mathbb{N}^{n+1}} \hat{f}(k, q) e^{ir_{kq}s} U_q(s, y) &= \sum_{(k,q) \in \mathbb{N}^{n+1}} \hat{f}(k, q) e^{ir_{kq}s} U_q(s, \sqrt{r_{kq}} y).
\end{aligned}$$

By theorem (2.49) we then have, at least formally,

$$(3.3) \quad W_\gamma^{-1} T^{-1} \Delta T W_\gamma \sim \mathcal{L}^2 + f_o(I_{\gamma 1}, \dots, I_{\gamma n}) + \frac{f_1(I_{\gamma 1}, \dots, I_{\gamma n})}{\mathcal{L}} + \dots$$

The purpose of this section is to make this equivalence precise. Since it is independent of the model we will carry out the proof in the basic model and use the weightless Fermi normal coordinates (s, x) . For the sake of notational simplicity the \tilde{W} 's and \tilde{Q} 's, transferred back to the model in this way, will be written as W 's and Q 's. Also in place of the r_{kq}^{-1} 's we will use the weightless h_{kq} 's with $h_{kq}^{-1} = (2\pi k + \sum_{j=1}^n (q_j + \frac{1}{2})\alpha_j) = Lr_{kq}$.

(3.4) **Proposition** $TW_\gamma T^{-1}$ is a (standard) Fourier integral operator, well-defined and invertible on the microlocal neighborhood (0.1) in $T^*(S_L^1 \times \mathbb{R}^n)$.

Sketch of Proof:

To simplify, we first consider the unitarily equivalent operator $\tilde{T}\tilde{W}\tilde{T}^{-1}$ in the microlocal neighborhood (0.1) in the twisted model, with

$$\begin{aligned}
(3.5) \quad \tilde{W} : \mathcal{H}_\alpha &\rightarrow \mathcal{H}_\alpha \\
\tilde{W}(e^{i\frac{s}{Lh_{kq}}} \gamma_q) &:= e^{i\frac{s}{Lh_{kq}}} W_{h_{kq}} \gamma_q,
\end{aligned}$$

and with \tilde{T} the dilation operator like (3.2) relative to the basis $e^{i\frac{s}{Lh_{kq}}} \gamma_q$. We then factor $\tilde{T}\tilde{W}\tilde{T}^{-1}$ as the product $\tilde{T}\tilde{W}\tilde{T}^{-1} = j^* \tilde{T} V \tilde{T}^{-1}$ where:

$$\begin{aligned}
(3.6) \quad V : \mathcal{H}_\alpha &\rightarrow L_{loc}^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n) \\
V &= \Pi_{j=o}^\infty \exp[iD_s^{-\frac{i}{2}} Q_{\frac{i}{2}}(s', y, D_y)]
\end{aligned}$$

that is,

$$V e^{i\frac{s}{Lh_{kq}}} \gamma_q(x) := e^{i\frac{s}{Lh_{kq}}} W_{h_{kq}}(s', x, D_x) \gamma_q(x),$$

and where

$$\begin{aligned}
(3.7) \quad j^* : C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n) &\rightarrow C^\infty(\mathbb{R} \times \mathbb{R}^n) \\
j^* f(s, x) &= f(s, s, x)
\end{aligned}$$

is the pullback under the partial diagonal embedding. We note that the variable s' in V occurs essentially as a parameter, so we can (and often will) regard V as a one parameter family of operators $V_{s'}$ on \mathcal{H}_α .

It is not clear that the infinite product in (3.6) is well-defined, nor what kind of operators the factors are. On the second point, we note that D_s is elliptic in the set (0.1) in $T^*(S_L^1 \times \mathbb{R}^n)$ and that, as an operator-valued function of s' ,

$$(3.8) \quad \tilde{T} D_s^{-\frac{j}{2}} Q_{\frac{j}{2}}(s', x, D_x) \tilde{T}^{-1} \in C^\infty(S_L^1, \Psi^1(S_L^1 \times \mathbb{R}^n)).$$

Here of course $C^\infty(S_L^1, \mathcal{A})$ denotes the smooth functions of s' with values in the algebra \mathcal{A} ; to simplify the notation we do not indicate the possible twisting. Indeed, we first observe that conjugation by \tilde{T} converts the mixed polyhomogeneous-isotropic algebra $\Psi^*(S_L^1) \otimes \mathcal{W}^*$ into the pure polyhomogeneous algebra $\Psi^*(S_L^1) \otimes \Psi^*(\mathbb{R}^n)$. This essentially follows from the fact that

$$\tilde{T} a^w(x, D_x) \tilde{T}^{-1} = a^w(D_s^{\frac{1}{2}} x, D_s^{-\frac{1}{2}} D_x)$$

(see [G.2] or [CV] for further details). Obviously, $\tilde{T} D_s \tilde{T}^{-1} \in \Psi^1(S_L^1 \times \mathbb{R}^n)$, and by (2.44(vi)) we also have

$$\tilde{T} Q_{\frac{j}{2}} \tilde{T}^* \in C^\infty(S_L^1, \Psi^{j+1}(S^1 \times \mathbb{R}^n)).$$

A simple calculation shows further that $\tilde{T} D_s^{-\frac{j}{2}} Q_{\frac{j}{2}}(s', x, D_x) \tilde{T}^{-1}$ is a one-parameter family of pseudodifferential operators of real principal type over $S_L^1 \times \mathbb{R}^n$ whose principal symbols have nowhere radial Hamilton vector field. It follows in a well-known way that these operators are microlocally conjugate to D_1 (see [Ho IV, Proposition 26.1.2]) and hence that their exponentials give a smooth one-parameter family of Fourier Integral operators. Consequently, after conjugation with \tilde{T} , the factors in (3.6) are for each $s' \in S^1$ microlocally well-defined Fourier Integral operators in the neighborhood (0.1), with smooth dependence on s' . As for the infinite product, we note that the principal symbol of $\tilde{T} D_s^{-\frac{j}{2}} Q_{\frac{j}{2}} \tilde{T}^{-1}$ has the form $\sigma^{-\frac{j}{2}} q_{\frac{j}{2}}(s', \sqrt{\sigma} y, \frac{1}{\sqrt{\sigma}} \eta)$, where $q_{\frac{j}{2}}$ is a homogenous polynomial in (y, η) of degree $j+2$ (2.36(iv)). Its exponential can therefore be constructed microlocally in (0.1) as a one parameter family of Fourier integral operators over $S_L^1 \times \mathbb{R}^n$ with phase functions of the form

$$\Psi_{\frac{j}{2}}(s', s, s'' \sigma, x, x'', \xi) = (s - s'') \sigma + \langle x - x'', \xi \rangle + \psi_{\frac{j}{2}}(s', s, s'', \sigma, x, x'', \xi)$$

with

$$(3.9b) \quad \psi_{\frac{j}{2}} \in O_{j+2} S_{cl}^1$$

Here, S_{cl}^1 denotes the space of classical symbols of first order in (0.1) and O_k denotes the elements which vanish to order k along $(x, \xi) = (0, 0)$. (Henceforth we will use the symbol O_k for objects of any kind which vanish to order k on this set.) It follows that for fixed s' and sufficiently small ϵ , the phase parametrizes the graph of a homogeneous canonical transformation of $T^*(S_L^1 \times \mathbb{R}^n)$ which is C^{j+1} -close to the identity. Any finite product $\Pi_{j=0}^N$ in (3.7) is therefore a clean composition and the phase Ψ_N of the product parametrizes the corresponding composition χ_N of canonical transformations. By (3.9b), we have $\chi_{N+1} - \chi_N \in O_{N+1}$ and so there exists a smooth one parameter family homogeneous canonical transformations $\chi_{s'}$, equal to the identity along $(x, \xi) = (0, 0)$ such that $\chi_{s'} - \chi_{s', N} \in O_{N+1}$ and with a generating function ψ_∞ satisfying $\psi_\infty - \psi_N \in O_N$ (cf. [Sj, Proposition 1.3]). Regarding the amplitudes, the situation is of a similar kind. Denoting the amplitude of $\tilde{T} \exp(D_s^{-\frac{j}{2}} Q_{\frac{j}{2}}) \tilde{T}^{-1}$ by $a_{\frac{j}{2}}(s', s, \sigma, y\eta)$ we have

$$(3.10) \quad a_{\frac{j}{2}} \equiv 1 \mod O_{j+2} S^o.$$

It follows that the amplitudes a_N of the products $\Pi_{j=0}^N$ satisfy $a_{N+1} - a_N \in O_{N+1} S^o$ and hence there exists an element a_∞ satisfying $a_\infty - a_N \in O_{N+1} S^o$. The pair (a_∞, ψ_∞) then determine a one-parameter family of Fourier integral operators of order zero which agrees with $\Pi_{j=0}^N$ up to an error which also vanishes to order N . Finally, composing with j^* just sets $s' = s$ in the kernel, which visibly remains Fourier integral, with phase function parametrizing the graph of a canonical transformation χ . Invertibility then follows from the fact that $\tilde{W}^* W$ is a pseudodifferential operator with principal symbol the square of $j^*(\sigma_V)$, which is easily seen to equal 1.

The proposition follows by expressing

$$TW_\gamma T^{-1} = T\mu(a)^* \tilde{T}^{-1} \tilde{T} W \tilde{T}^{-1} \tilde{T} \mu(a) T^{-1}$$

and noting that $\tilde{T}\mu(a)T^{-1}$ is also a standard Fourier Integral operator. \square

We now give the proof of the quantum normal form Theorem B for $\sqrt{\Delta}$, stated in an equivalent form in terms of W_γ . As in the introduction, the notation $A \equiv B$ means that the complete (Weyl) symbol of $A - B$ vanishes to infinite order at γ . We also use the notation $O_j \Psi^m$ for pseudodifferential operators of order m whose Weyl symbols vanish to order j at $(y, \eta) = (0, 0)$. Here, pseudodifferential operator could mean of the standard polyhomogeneous kind, or of the mixed polyhomogeneous-isotropic kind as in $\Psi^k(S_L^1) \otimes \mathcal{W}^l$, in which case the total order is defined to be $m = k + l$. To simplify notation, we will denote the space of mixed operators of order m by $\Psi_{mx}^m(S_L^1 \times \mathbb{R}^n)$.

(3.11) **Theorem B** *Let $TW_\gamma T^{-1}$ be the Fourier Integral operator of Proposition (3.4), defined over a conic neighborhood of $R^+ \gamma$ in $T^*(S_L^1 \times \mathbb{R}^n)$. Then:*

$$W_\gamma^{-1} T^{-1} \sqrt{\Delta} T W_\gamma \equiv P_1(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) + P_o(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) + \dots,$$

where

$$(3.14b) \quad P_1(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) \equiv \mathcal{L} + \frac{p_1^{[2]}(I_{\gamma 1}, \dots, I_{\gamma n})}{L\mathcal{L}} + \frac{p_2^{[3]}(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^2} + \dots$$

$$P_{-m}(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) \equiv \sum_{k=m}^{\infty} \frac{p_k^{[k-m]}(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^j}$$

with $p_k^{[k-m]}$, for $m=-1, 0, 1, \dots$, homogenous of degree $k-m$ in the variables $(I_{\gamma 1}, \dots, I_{\gamma n})$ and of weight -1 .

Proof:

As a semi-classical expansion in the “parameter” $h = \frac{1}{L\mathcal{L}}$, (3.3) may be rewritten in terms of $\sqrt{\Delta}$:

$$(3.12) \quad W_\gamma^{-1} T^{-1} \sqrt{\Delta} T W_\gamma \sim \mathcal{L} + \frac{p_1(I_{\gamma 1}, \dots, I_{\gamma n})}{L\mathcal{L}} + \frac{p_2(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^2} + \dots$$

From the fact that the numerators $f_j(I_{\gamma 1}, \dots, I_{\gamma n})$ in (3.3) are polynomials of degree $j+2$ (2.36viii) and of weight -2 , the numerators $p_k(I_{\gamma 1}, \dots, I_{\gamma n})$ are easily seen to be polynomials of degree $k+1$ and of weight -1 . Hence they may be expanded in homogeneous terms

$$(3.13) \quad p_k = p_k^{[k+1]} + p_k^{[k]} + \dots p_k^{[o]},$$

with $p_k^{[j]}$ the term of degree j and still of weight -1 . The right side of (3.12) can then be expressed as a sum of homogeneous operators:

$$(3.14a) \quad P_1(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) + P_o(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) + \dots$$

with

$$(3.14b) \quad P_1(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) \equiv \mathcal{L} + \frac{p_1^{[2]}(I_{\gamma 1}, \dots, I_{\gamma n})}{L\mathcal{L}} + \frac{p_2^{[3]}(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^2} + \dots$$

$$P_{-m}(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) \equiv \sum_{k=m}^{\infty} \frac{p_k^{[k-m]}(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^k}.$$

The remainder term in (3.12) can be described as follows:

$$(3.15) \quad W_\gamma^{-1} T^{-1} \sqrt{\Delta} T W_\gamma - [\mathcal{L} + \frac{p_1(I_{\gamma 1}, \dots, I_{\gamma n})}{L\mathcal{L}} + \frac{p_2(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^2} + \dots + \frac{p_m(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^m}]$$

$$\in \oplus_{k=o}^{m+1} O_{2(m+1-k)} \Psi_{mx}^{1-k}(S_L^1 \times \mathbb{R}^n).$$

The remainder terms in (3.14b) are given by:

$$(3.16a) \quad P_1(\mathcal{L}, I_{\gamma 1}, \dots, I_{\gamma n}) - [\mathcal{L} + \frac{p_1^{[2]}(I_{\gamma 1}, \dots, I_{\gamma n})}{L\mathcal{L}} + \dots + \frac{p_N^{[N+1]}(I_{\gamma 1}, \dots, I_{\gamma n})}{(L\mathcal{L})^k}]$$

$$\in O_{2(N+2)}\Psi_{mx}^1(S_L^1 \times \mathbb{R}^n)$$

$$(3.16b) \quad P_{-m}(\mathcal{L}, I_{\gamma_1}, \dots, I_{\gamma_n}) - \sum_{k=m}^N \frac{p_k^{[k-m]}(I_{\gamma_1}, \dots, I_{\gamma_n})}{(L\mathcal{L})^k} \in O_{2(N+1-m)}\Psi_{mx}^{-m}(S_L^1 \times \mathbb{R}^n).$$

Hence the expansion (3.12) is also asymptotic in the sense of \equiv . For the statement of Theorem B in the introduction, we only need to conjugate under $\mu(\tilde{a})$. \square

Remark The remainder in (3.15) could be equivalently described by saying that its complete symbol lies in the symbol class $S_{cl}^1 \cap S^{1,2(m+1)}$ of Boutet-de-Monvel [BM].

4. RESIDUES AND WAVE INVARIANTS: PROOF OF THEOREM C

The aim now is to use the normal form of $\sqrt{\Delta}$ near γ to calculate the wave invariants $a_{\gamma k}$ associated to γ . In terms of the model (see the statement of Theorem B in the introduction) we may write the normal form as

$$(4.1) \quad \mathcal{D} = \overline{\psi}(I_1, \dots, I_n)[\mathcal{D}_N + \mathcal{B}_N]$$

with

$$\mathcal{D}_N := \mathcal{R} + \frac{p_1(I_1, \dots, I_n)}{L\mathcal{R}} + \dots + \frac{p_N(I_1, \dots, I_n)}{(L\mathcal{R})^N}$$

and with

$$\mathcal{B}_N \in \oplus_{k=0}^{N+1} O_{2(N+1-k)}\Psi_{mx}^{1-k}(S_L^1 \times \mathbb{R}^n).$$

Our first observation is that we can drop a sufficiently high remainder term \mathcal{B}_N in the calculation of a given wave invariant. We prove this by working out, roughly, which parts of the complete symbol of a general first order pseudodifferential operator P of real principal type contribute to the wave invariant of a given order associated to a non-degenerate closed bicharacteristic γ .

In the following proposition, (s, y) will denote any coordinates in a tubular neighborhood U of the projection of γ to M with the property that the equation for $\mathbb{R}^+\gamma$ in the associated symplectic coordinates (s, σ, y, η) is $y = \eta = 0$. Also, Op we will denote a fixed quantization of symbols in a conic neighborhood $(0,1)$ of $\mathbb{R}^+\gamma$ to pseudodifferential operators, and $p(s, \sigma, y, \eta) \sim p_1 + p_0 + \dots$ will denote the complete symbol of P . The Taylor expansion of p_j at $\mathbb{R}^+\gamma$ will be written in the form,

$$p_j(s, \sigma, y, \eta) = \sigma^j p_j(s, 1, y, \frac{\eta}{\sigma}) = \sigma^j (p_j^{[0]} + p_j^{[1]} + \dots)$$

with $p_j^{[m]}(s, 1, y, \frac{\eta}{\sigma})$ homogeneous of degree m in $(y, \frac{\eta}{\sigma})$. We set $P_j := Op(p_j)$, $P_j^{[m]} := Op(p_j^{[m]})$ and $P_j^{\leq N} = \sum_{m \leq N} P_j^{[m]}$. Finally, we will denote by $\tau_{\gamma k}(P)$ the coefficient of $(t - L + i0)^k \log(t - L + i0)$ (or of $(t - L + i0)^{-1}$ in the case of $k = -1$) in the singularity expansion of the microlocal unitary group e^{itP} near γ .

(4.2) Proposition

$$\tau_{\gamma k}(P) = \tau_{\gamma k}(P_1^{\leq 2(k+2)} + P_0^{\leq 2(k+1)} + \dots P_{-k-1}^o).$$

Proof: As mentioned several times, the wave invariant $\tau_{\gamma k}(P)$ is given by the non-commutative residue

$$(4.3) \quad \tau_{\gamma k}(P) = \text{res} D_t^k e^{itP}|_{t=L} = \text{res} P^k e^{itP}|_{t=L}$$

(see [Z.1] or [G.2]). We first describe how (4.3) leads to a local formula for $\tau_{\gamma k}(P)$ in terms of the jets of the amplitudes and phases of a microlocal parametrix

$$(4.4) \quad F_{\gamma}(t, x, x') = \int_{\mathbb{R}^n} e^{i\phi(t, x, x', \theta)} a(t, x, x', \theta) \psi(x, \theta) d\theta$$

for e^{itP} near $\{L\} \times R^+\gamma \times R^+\gamma$. The remainder of the proof will connect this data to the complete symbol of P .

The amplitude a in (4.4) is a classical symbol, hence has the expansion $a \sim \sum_{j=0}^{\infty} a_{-j}$ with a_{-j} homogeneous of degree $-j$ for $|\theta| \geq 1$. Since the residue or wave invariant depends only on the wave kernel

in a microlocal neighborhood of $\{L\} \times \gamma \times \gamma$ it is unchanged by the insertion homogeneous cut-off ψ , supported in (0.1) and equal to one on a smaller conic neighborhood of γ . The phase $\phi(t, x, x', \theta)$ parametrizes a piece of the graph of the Hamilton flow $\exp t H_{p_1}$ of p_1 near $\{L\} \times \gamma$, and may be assumed to consist of only n phase variables. Writing $x = (s, y)$ and $\theta = (\sigma, \eta)$ as above and choosing ψ of the form $\psi(\frac{\eta}{\sigma})$, we get

$$(4.5) \quad \tau_{\gamma k}(P) = \text{Res}_{s=0} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n-1}} \int_U a^{(k)}(L, s, y, s, y, \sigma, \eta) \psi\left(\frac{\eta}{\sigma}\right) e^{i\phi(L, s, y, s, y, \sigma, \eta)} |\sigma|^{-s} d\sigma d\eta dv(s, y)$$

where for certain universal constants $C_{\alpha\beta}$,

$$(4.5.1) \quad a^{(k)} := \sum_{|\alpha|+|\beta|=k} C_{\alpha\beta} (D_t^{\alpha_1} \phi)^{\beta_1} \dots (D_t^{\alpha_n} \phi)^{\beta_n} D_t^{\alpha_{n+1}} a.$$

We note here that the residue can be calculated using any gauging of the trace, in particular by powers $|\sigma|^{-s}$ of the elliptic symbol σ in (0.1). Changing variables to $\tilde{\eta} := \frac{\eta}{\sigma}$ and denoting by $a_{k-j}^{(k)}$ the term of order $k-j$ in the polyhomogenous expansion of the k th order amplitude $a^{(k)}$, and by B_ϵ the ball of radius ϵ in $\mathbb{R}_{\tilde{\eta}}^n$, we get

$$(4.6) \quad \begin{aligned} \tau_{\gamma k}(P) &= \sum_{j=0}^{\infty} \text{Res}_{s=0} \int_U \int_{\mathbb{R}^+} \int_{B_\epsilon} e^{i\sigma\phi(L, s, y, s, y, 1, \tilde{\eta})} a_{k-j}^{(k)}(L, s, y, s, y, 1, \tilde{\eta}) \psi(s, y, \tilde{\eta}) \sigma^{n+k-1-j-s} d\sigma d\tilde{\eta} dv(s, y) \\ &= \sum_{j=0}^{\infty} \int_U \int_{B_\epsilon} \psi a_j^{(k)} (\phi(L, s, y, s, y, 1, \tilde{\eta}) + i0)^{s+j-n-k} d\tilde{\eta} dv(s, y). \end{aligned}$$

The non-degeneracy of γ as a closed geodesic implies its non-degeneracy as the critical manifold of $\phi(s, y, s, y, 1, \tilde{\eta})$. Hence we have

$$\phi(L, s, y, s, y, 1, \tilde{\eta}) = \phi(L, s, 0, s, 0, 1, 0) + Q_s(y, \tilde{\eta}) + g(s, y, \tilde{\eta})$$

with Q a non-degenerate symmetric bilinear form and with $g = 0_s(|(y, \tilde{\eta})|^3)$. Expanding

$$(Q_s(y, \tilde{\eta}) + i0 + g)^\lambda = \sum_{p=0}^{\infty} C_{(\lambda, p)} (Q_s(y, \tilde{\eta}) + i0)^{\lambda-p} g^p$$

we get that

$$(4.7) \quad \tau_{\gamma k}(P) = \sum_{j, p=0}^{\infty} C_{(s+j-n-k, p)} \text{Res}_{s=0} \int_{\gamma} \int_{|y| < \epsilon} \int_{B_\epsilon} (Q_s(y, \tilde{\eta}) + i0)^{s+j-n-k-p} (g^p a_{k-j}^{(k)} \psi J(s, y)) ds dy d\tilde{\eta}$$

with $J(s, y)$ the volume density and with binomial coefficients $C_{(\lambda, p)}$. The family of distributions $(Q_s(y, \tilde{\eta}) + i0)^\lambda$ is meromorphic with simple poles at the points $\lambda = -(n-1) - r, r = 0, 1, 2, \dots$ and with residue $C_r(Q_s^{-1}(D_y, D_\eta))^r \delta(y, \eta)$ for certain constants C_r (see, [G.S., p.276]). Here we have dropped the tilde in the notation for η . For $s = 0$ there are possible poles when $k + p + 1 - j \geq 0$ with residues

$$(4.8) \quad C_r C_{(j-n-k, p)} \int_{\gamma} Q_s^{-1}(D_y, D_\eta)^{k+p+1-j} (g^p a_{k-j}^{(k)} J)|_{(y=\eta=0)} ds.$$

However, the residue vanishes unless $2(k + p + 1 - j) \geq 3p$ since g^p vanishes to order $3p$, constraining the sum in (4.6) to $2j + p \leq 2(k + 1)$. Hence we have, with new constants C_{jkp} ,

$$(4.9) \quad \tau_{\gamma k}(P) = \sum_{j, p: 2j+p \leq 2(k+1)} C_{jkp} \int_{\gamma} Q_s^{-1}(D_y, D_\eta)^{k+p+1-j} (g^p a_{k-j}^{(k)} J)|_{(y, \eta)=(0,0)} ds.$$

It follows first that $j \leq k + 1$, hence that only the terms $a_k^{(k)}, \dots, a_{-1}^{(k)}$ in the amplitude $a^{(k)}$ contribute to (4.9). To determine the corresponding range of terms a_j in the amplitude a , we observe from (4.5.1) that

$$(4.10) \quad a_{k-j}^{(k)} := \sum_{|\alpha|+|\beta|=k} C_{\alpha\beta} (D_t^{\alpha_1} \phi)^{\beta_1} \dots (D_t^{\alpha_n} \phi)^{\beta_n} D_t^{\alpha_{n+1}} a_{k-j-|\beta|},$$

hence that only the terms $a_o, a_{-1}, \dots, a_{-k-1}$ contribute to (4.9).

The same kind of calculation determines $\text{res} A e^{itP}|_{t=L}$, for any k th order pseudodifferential operator A supported microlocally in (0.1) in terms of the complete symbol of A and the phases and amplitudes in (4.3). In particular, we see that $\text{res} D_t^k A e^{itP}|_{t=L} = 0$ if A has order $-(k+2)$. This implies:

$$(4.11) \quad \text{res} D_t^k e^{itP} = \text{res} D_t^k e^{it(P_1 + \dots + P_{-(k+1)})}$$

since $e^{itP} = A_k(t) e^{it(P_1 + \dots + P_{-(k+1)})}$ with $A_k(t) - I \in \Psi^{-k-2}$. Indeed, with $H_k = P_1 + \dots + P_{-(k+1)}$, $V_k = P - H_k$, $\tilde{B}(t) := e^{-itH_k} B(t) e^{itH_k}$, we have

$$D_t \tilde{A}_k = \tilde{V}_k(t) \tilde{A}_k(t), \quad \tilde{A}_k(0) = I.$$

Thus $\tilde{A}_k(t)$ is the time-ordered exponential of a pseudodifferential operator of order $-k-2$, hence equal to I modulo Ψ^{-k-2} ; consequently so is $A_k(t)$.

We next consider which jets $j_\gamma^m a_{-i}$ of a_o, \dots, a_{-k-1} and $j_\gamma^m \phi$ of ϕ contribute to (4.9). Here by $j_\gamma^m a_{-i}$ we mean the Taylor polynomial of degree m of $a_{-i}(t, s, y, s, y, \sigma, \eta)$ in the (y, η) variables for t near L . From (4.9-10) it is evident that the maximum number of (y, η) derivatives on a_{-i} occurs when $p = 0$ in (4.9) and when $|\beta| = k, k - j - |\beta| = -i$ in (4.10). It follows that at most $2(k+1-i)$ derivatives fall on a_{-i} in (4.9). Also, the maximum number of (y, η) derivatives on the phase occurs in terms where a factor of g is differentiated the maximum number of times; namely in terms with $p = 1$ and with $j = 0$, in which the phase is differentiated $2(k+2)$ times. Hence

$$(4.12) \quad \tau_{\gamma k}(P) \text{ depends only on } j_\gamma^{2(k+1)} a_o, j_\gamma^{2k} a_{-1}, \dots, j_\gamma^o a_{-(k+1)}; j_\gamma^{2(k+2)} \phi.$$

To give bounds on the jets of the terms $p_j (j = 1, 0, \dots, -(k+1))$ in the symbol of P which contribute to the jets $j_\gamma^{2(k+1-i)} a_{-i}$, we now have to consider some details of the construction of the parametrix (4.3). We first recall that the amplitudes are obtained by solving transport equations of the form

$$(4.13) \quad D_t a_{-j} = \sum_{m=0}^j [\mathcal{P}_{1,m+1} a_{-j+m} + \sum_{\nu=0}^{j-m} \mathcal{P}_{-\nu,m} a_{-j+m+\nu}]$$

where $\mathcal{P}_{-\nu,m} = \mathcal{P}_{-\nu,m}(\phi, s, y, D_s, D_y)$ is a differential operator of order $\leq m$ obtained from $Op(p_{-\nu})$ as follows:

$$(4.14.1) \quad e^{-i\rho\phi} Op(p_{-\nu}) e^{i\rho\phi} \sim \sum_{m=0}^{\infty} \rho^{-\nu-m} \mathcal{P}_{-\nu,m}$$

ϕ being the phase in the microlocal parametrix (see [Tr, Ch.VI,(5.28)]). Consistently with (4.11), only the terms $p_1, p_o, \dots, p_{-(k+1)}$ in the symbol of P contribute to the transport equation for $a_{-i} (i = o, \dots, -(k+1))$. To determine which jet of $p_{-\nu}$ is involved in $\mathcal{P}_{-\nu,m}$, we further recall [loc.cit., Theorem 3.1] that the expansion in (4.14.1) is obtained by re-arranging terms in the expansion

$$(4.14.2) \quad e^{-i\rho\phi} Op(p_{-\nu}) e^{i\rho\phi} u \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} p_{-\nu}(s, y, \rho d\phi) \mathcal{N}_{\alpha}(\phi; \rho, D_s, D_y) u$$

with

$$\begin{aligned} \mathcal{N}_{\alpha}(\phi; \rho, D_x) u &= D_x^{\alpha} e^{i\phi_2(x, x')} u(x')|_{x=x'} \\ \phi_2(x, x') &= \phi(x) - \phi(x') - \langle x - x', \nabla \phi(x) \rangle. \end{aligned}$$

In view of the fact that \mathcal{N}_{α} is a polynomial in ρ of degree $\frac{|\alpha|}{2}$, we see that $|\alpha| \leq 2m$ in any term of (4.14.2) contributing to $\mathcal{P}_{-\nu,m}$. Hence $\mathcal{P}_{-\nu,m}$ involves at most $2m$ derivatives of $p_{-\nu}$, and so the transport equation for a_{-i} involves at most $2(i-\nu)$ derivatives of $p_{-\nu}$ ($\nu = -1, 0, \dots, i$).

To draw these kinds of conclusions about the amplitudes a_{-i} themselves, as opposed to the coefficients in their transport equations, we have to take into account the initial conditions in the transport equations. Unfortunately, the defining initial conditions occur at $t = 0$ while we are interested in the long time behaviour at $t = L$. Were there no conjugate pairs along geodesics near γ , the transport equations could be solved up to $t = L$ and since the initial conditions can be taken in the form $a_o|_{t=0} \equiv 1, a_{-i}|_{t=0} \equiv 0$ we could conclude that the a_{-i} 's are integrals of polynomials in at most $2(i-\nu)$ derivatives of $p_{-\nu}$ and in at most $2(i+1)$ derivatives of the phase (coming from the \mathcal{N}_{α} 's). However, in the case of elliptic closed geodesics, there will

always be conjugate pairs for L sufficiently large and we cannot construct the parametrix so simply. Rather we will use the group property $U(L) = U(L/N)^N$ with N sufficiently large that a parametrix for $U(L/N)$ can be constructed by the geometric optics method. We then have to eliminate all but n phase variables in the power, which will lead to crude but serviceable bounds on the order of the jets.

We therefore begin with the construction of a short time parametrix $F_{\delta k}$, valid for $|t| < \delta$, of the form (4.4) with phase $\phi(t, x, x', \theta) = S(t, x, \theta) - \langle x', \theta \rangle$, satisfying $\partial_t S + p_1(x, d_x S) = 0$, $S|_{t=0} = \langle x, \theta \rangle$, and with amplitude $a \sim \sum_{j=0}^{k+1} a_{-j}$ satisfying (4.13) on $|t| \leq \delta$ and with initial conditions $a_o|_{t=0} = 1$, $a_{-j}|_{t=0} = 0$ ($j > 0$). By the observations above, we have that $a_{\delta, -i}(t, x, x', \theta)$ involves at most $2(i - \nu)$ derivatives of $p_{-\nu}$ ($\nu \leq i$) for t in this interval. As is well-known, we then have $e^{itP} - F_{\delta k}(t) = R_{ok}(t)$ with the remainder $R_{\delta k}(t)$ a Fourier integral operator of order $-(k+2)$ associated to the graph of $\text{expt}H_{p_1}$ for $|t| < \delta$. Since $F_{\delta k}$ is of order zero, it follows that for any N and for $|t| < \delta$, $e^{itNP} \equiv F_{\delta k}^N(t)$ modulo Fourier integral operators of order $-k-2$ in this class. Hence, the desired parametrix (4.4) can be constructed by choosing N so that $L/N < \delta$ and by eliminating all but n phase variables in $F_{\delta k}^N$ by the stationary phase method. Just as in (4.11) the remainder terms with a factor of $R_{\delta k}$ will not contribute to $\tau_{\gamma k}(P)$, nor will terms a_{-i} in the final amplitude with $i > k+1$.

To complete the proof of the proposition, we have to count the number of (y, η) -derivatives of $p_{-\nu}$ which appear in the terms

$$(4.15.1) \quad D_t^{k-|\beta|} Q_s^{-1}(D_y, D_\eta)^{k+1-j} \hat{a}_{k-j-|\beta|}|_{(y, \eta)=(0,0)},$$

and the number of derivatives of the final phase (hence of p_1) that occur in the terms

$$(4.15.2) \quad Q_s^{-1}(D_y, D_\eta)^{k+p+1-j} g^p (D_t^{\alpha_1} \phi)^{\beta_1} \dots (D_t^{\alpha_n} \phi)^{\beta_n} D_t^{\alpha_{n+1}} \hat{a}_{k-j-|\beta|}|_{(y, \eta)=(0,0)}$$

with \hat{a}_{-i} the terms in the final amplitude. We start from the fact that, in an obvious notation,

$$(4.16) \quad F_{k\delta}^N = \int_{\mathbb{R}^{nN}} \int_{M^{N-1}} e^{i(\phi_1 + \dots + \phi_N)} \sum_{(i_1, \dots, i_N) \in \mathbf{N}^N} a_{-i_1} \dots a_{-i_N} d\theta_1 \dots d\theta_N dv(x_1) \dots dv(x_{N-1})$$

and that the elimination of all but n phase variables by the stationary phase method gives essentially the same formulae for the final amplitudes \hat{a}_{-i} as in the case of non-homogeneous Lagrangean distributions of the form $\int u(x, y) e^{i\omega f(x, y)} dx$ with $f(x, y)$ a real-valued function defined near $(0, 0)$, with $f'_x(0, 0) = 0$, and with $f''_{xx}(0, 0)$ non-singular. In this situation, we have (see [HoI, Theorems 7.7.5-6])

$$\int u(x, y) e^{i\omega f(x, y)} dx = C \frac{e^{i\omega f(x(y), y)}}{|\det(\omega f''_{xx}(x(y), y))|^{-\frac{1}{2}}} \sum_{j + \frac{n}{2} < M} L_{f, j, y} u(x(y), y) \omega^{-j}$$

modulo terms of order $O(\omega^{-(M)})$, with $L_{f, j, y}$ a differential operator of order $2j$ whose coefficients involve at $2j$ derivatives of f'' . It follows that \hat{a}_{-i} has the form (with $I = (i_1, \dots, i_N)$, $\Phi = \phi_1 + \dots + \phi_N$)

$$\sum_{I, j: |I|+j=i} L_{\Phi, j} a_{-i_1} \dots a_{-i_N}.$$

Since also a_{-i} has the form

$$F_i(p_1, Dp_1, \dots, D^{2(i+1)}p_1, p_o, \dots, D^{2i}p_o, \dots, p_{-i}; \phi, \dots, D^{2i}\phi)$$

with F_i a multiple integral of polynomials, and with D^k some differential operators of degree k , we see that also \hat{a}_{-i} has the form

$$\mathcal{F}_i(p_1, \dots, D^{2(i+1)}p_o, \dots, D^{2i}p_o, \dots, D^2p_{-i+1}, p_{-i}; \phi, \dots, D^{2(i+1)}\phi).$$

Hence as regards number of derivatives of p_1, \dots, p_{-i}, ϕ , the \hat{a}_{-i} 's behave exactly as the a_i 's, that is, involve at most $2(i - \nu)$ derivatives of $p_{-\nu}$. Hence (4.9)-(4.10) apply to the final amplitudes, and we conclude that

$$\tau_{\gamma k}(P) \text{ depends only on } j_\gamma^{2(k+2)}p_1, j_\gamma^{2(k+1)}p_o, \dots, j_\gamma^o p_{-(k+1)}; j_\gamma^{2(k+2)}\phi.$$

Since $j^{2(k+2)}\phi$ depends only on $j^{2(k+1)}p_1$, the proof of the proposition is complete. \square

(4.16) **Corollary**

$$\tau_{\gamma k}(\mathcal{D}) = \tau_{\gamma k}(\mathcal{D}_{k+1}).$$

The following Lemma combines the previous results in a form applicable to the calculation of the wave invariants. The notation $TrAP^{-s}$ (with $P \in \Psi^1$ elliptic) is short for the zeta function obtained by meromorphic continuation of the trace from $Res \gg 0$ to \mathbb{C} .

(4.17) **Lemma**

$$\tau_{\gamma k}(\sqrt{\Delta}) = Res_{s=0} D_t^k Tr \bar{\psi}(\mathcal{R}, I_1, \dots, I_n) e^{it\mathcal{D}_{k+1}} \mathcal{R}^{-s} |_{t=L}$$

Proof Since \mathcal{R} is elliptic in the essential support of $\bar{\psi}$, the trace on the right side is well defined and has a meromorphic continuation to \mathbb{C} (cf. [G.2], [Z.1,5]). The residue is of course $\tau_{\gamma k}(\mathcal{D})$ by the previous proposition. Hence it suffices to show that

$$(4.18) \quad \tau_{\gamma k}(\sqrt{\Delta}) = \tau_{\gamma k}(\mathcal{D}).$$

This however follows from the previous proposition combined with Theorem B: Indeed, the proposition shows that $\tau_{\gamma k}(A) = \tau_{\gamma k}(B)$ if $A \equiv B$ in the sense of Theorem B. Also, $\tau_{\gamma k}(WAW_1) = \tau_{\gamma k}(A)$ if W_1 is a parametrix for W on the essential support of A , as follows from the tracial property $res WAW_1 = res W_1WA$ of the residue (see [Z.1,5] for instance). \square

We come now to the calculation of the wave invariants as residue traces of the normal form wave group. But we will simplify (4.17) further before evaluating the residue trace. First, we rewrite everything in terms of D_s , and H_α using (0.3-4). Since

$$\frac{p_\nu(I_1, \dots, I_n)}{(L\mathcal{R})^\nu} = \frac{p_\nu(I_1, \dots, I_n)}{(LD_s)^\nu} (I - \nu \frac{H_\alpha}{LD_s} + \frac{1}{2} \nu(\nu-1) (\frac{H_\alpha}{LD_s})^2 + \dots)$$

and since we can drop the $D_s^{-(k+1)+\nu} H_\alpha^{k+1-\nu}$ and higher terms by Proposition (4.2), \mathcal{D}_{k+1} can be written in the form

$$(4.19) \quad \mathcal{D}_{k+1} \equiv D_s + H_\alpha + \frac{\tilde{p}_1(I_1, \dots, I_n, L)}{LD_s} + \frac{\tilde{p}_2(I_1, \dots, I_n, L)}{(LD_s)^2} + \dots + \frac{\tilde{p}_{k+1}(I_1, \dots, I_n, L)}{(LD_s)^{k+1}}$$

modulo terms which make no contribution to $\tau_{\gamma k}$. Secondly, we can use LD_s rather than \mathcal{R} as the gauging elliptic operator in (4.17). To simplify the notation we will denote all but the first two terms on the right side of (4.19) by

$$(4.20) \quad \mathcal{P}_{k+1}(D_s, I_1, \dots, I_n, L) := \frac{\tilde{p}_1(I_1, \dots, I_n, L)}{LD_s} + \frac{\tilde{p}_2(I_1, \dots, I_n, L)}{(LD_s)^2} + \dots + \frac{\tilde{p}_{k+1}(I_1, \dots, I_n, L)}{(LD_s)^{k+1}}.$$

Then we have:

$$(4.21) \quad \tau_{\gamma k}(\sqrt{\Delta}) = Res_{z=0} Tr D_t^k \bar{\psi}(D_s, I_1, \dots, I_n) e^{it[\frac{1}{L}(2\pi LD_s + H_\alpha) + \mathcal{P}_{k+1}]} (LD_s)^{-z} |_{t=L}.$$

We can now give:

Proof of Theorem C: By (4.21) and the fact that $e^{2\pi i LD_s} \equiv I$ on $L^2(S_L^1)$ we get

$$(4.22) \quad a_{k\gamma} = \tau_{\gamma k}(\sqrt{\Delta}) = Res_{z=0} Tr [\frac{1}{L} 2(\pi LD_s + H_\alpha) + \mathcal{P}_{k+1}]^k e^{iH_\alpha} e^{iL\mathcal{P}_{k+1}} \bar{\psi}(\frac{I}{\epsilon D_s}) (LD_s)^{-z}$$

with

$$\bar{\psi}(\frac{I}{\epsilon D_s}) := \Pi_{j=1}^n \bar{\psi}(\frac{I_j}{\epsilon D_s}).$$

From the well-known spectra of D_s, I_1, \dots, I_n we can rewrite (4.22) as

$$(4.23) \quad Res_{z=0} \sum_{m=1}^{\infty} m^{-z} \left\{ \sum_{(q_1, \dots, q_n) \in \mathbb{N}^n} \left[\frac{1}{L} (2\pi m + \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j) + \mathcal{P}_{k+1}(m, q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}) \right]^k \cdot e^{iL\mathcal{P}_{k+1}(m, q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}, L)} \bar{\psi}(\frac{q}{\epsilon m}) e^{i \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j} \right\}.$$

Regarding

$$(4.24) \quad \sum_{(q_1, \dots, q_n) \in \mathbb{N}^n} \bar{\psi}\left(\frac{q}{\epsilon m}\right) e^{i \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j}$$

as a smooth function of $\alpha \in \mathbb{R}^n$, we can further rewrite (4.23) as

$$(4.25) \quad \text{Res}_{z=0} \sum_{m=1}^{\infty} m^{-z} \left\{ \left[\frac{1}{L} (2\pi m + \sum_{j=1}^n \alpha_j D_{\alpha_j}) + \mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L) \right]^k \right. \\ \left. e^{iL\mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L)} \sum_{(q_1, \dots, q_n) \in \mathbb{N}^n} \bar{\psi}\left(\frac{q}{\epsilon m}\right) e^{i \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j} \right\}.$$

Since $\bar{\psi}(\frac{q}{\epsilon m})$ is for each m a finitely supported function of q , we can also rewrite (4.24) as

$$(4.26) \quad \lim_{\delta \rightarrow 0} \prod_{j=1}^n \bar{\psi}\left(\frac{D_{\alpha_j}}{\epsilon m}\right) \sum_{q_j=0}^{\infty} e^{i \sum_{j=1}^n \alpha_j (q_j + \frac{1}{2}) - \delta} \\ = \lim_{\delta \rightarrow 0} \prod_{j=1}^n \bar{\psi}\left(\frac{D_{\alpha_j}}{\epsilon m}\right) \prod_{j=1}^n \frac{e^{i \frac{1}{2} \alpha_j}}{1 - e^{i(\alpha_j + i\delta)}}.$$

Recalling the definition of $T(\alpha)$ (0.12a), combining (4.23)-(4.26), and taking the limit as $\delta \rightarrow 0$, we get:

$$(4.27) \quad a_{\gamma k} = \text{Res}_{z=0} \sum_{m=1}^{\infty} m^{-z} \left\{ \left[\frac{1}{L} (2\pi m + \sum_{j=1}^n \alpha_j D_{\alpha_j}) + \mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L) \right]^k \right. \\ \left. e^{iL\mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L)} \prod_{j=1}^n \bar{\psi}\left(\frac{D_{\alpha_j}}{\epsilon m}\right) T(\alpha) \right\}.$$

Here, we use the fact that $T(\alpha)$ is a temperate distribution on \mathbb{R}^n with singular support on $\cup_{j=1}^n \mathbb{R} \times \mathbb{R} \cdots \times 2\pi\mathbb{Z} \cdots \times \mathbb{R}^n$ (the factor of \mathbb{Z} occurring in the j th position) and that the limit $\delta \rightarrow 0$ in (4.26) can be taken in \mathcal{S}' . Since the cut-off smooths out the singularity, the right side of (4.27) is a smooth function of α and can be evaluated at the special values of α determined by γ .

Since $\mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L)$ is a symbol of order -1 in m with coefficients given by polynomials in the operators D_{α_j} , we can expand the k th power in (4.27) as an operator-valued polyhomogeneous function of m . At least formally, we can also expand the exponential $e^{iL\mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n}, L)}$ in a power series and then expand each term in the power series as a polynomial in m^{-1} . Collecting powers of m , the right side of (4.27) can be put, at least formally, in the form

$$(4.28) \quad \text{Res}_{z=0} \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} m^{-z+k-j} \mathcal{F}_{k,k-j}(D_{\alpha}, L) \bar{\psi}\left(\frac{D_{\alpha}}{m\epsilon}\right) T(\alpha),$$

with $\mathcal{F}_{k,k-j}(D_{\alpha}, L)$ the coefficient of m^{k-j} in (4.27). To justify this manipulation, we have to deal with the remainder term in the Taylor expansion of the exponential. We have

$$e^{ix} = e_N(ix) + r_N(ix) \\ e_N(ix) = 1 + ix + \cdots + \frac{(ix)^N}{N!} \\ r_N(ix) = (ix)^{N+1} b_N(ix) \\ b_N(ix) = \int_0^1 \cdots \int_0^1 t_N^N t_{N-1}^{N-1} \cdots t_o^o e^{t_N t_{N-1} \cdots t_o ix} dt_o \cdots dt_N.$$

Hence we have

$$e^{iL\mathcal{P}_{k+1}} := e_N(iL\mathcal{P}_{k+1}) + (iL\mathcal{P}_{k+1})^{N+1} b_N(iL\mathcal{P}_{k+1})$$

with \mathcal{P}_{k+1} short for $\mathcal{P}_{k+1}(m, D_{\alpha_1}, \dots, D_{\alpha_n})$. The e_N term of course contributes a finite number of terms of the desired form (4.28). For the remainder, we expand $(iL\mathcal{P}_{k+1})^{N+1}$ as a polynomial in m^{-1} with coefficients given by operators $Q_{Np}(D_{\alpha_1}, \dots, D_{\alpha_n})$ and observe that each term has a factor of m^{-N-1} . For each such term, we remove the coefficient operator Q_{Np} from the sum \sum_m , leaving only the factor of b_N .

We then rewrite the resulting sum as a double sum \sum_{mq} as in (4.23), replacing all operators in D_{α_j} by their eigenvalues. Since $b_N(ix)$ is a bounded function and since each term of the resulting sum has at least the factor $m^{-z-N-1+k}$ (possibly multiplied by a negative power of m), we see that the remainder is a sum of terms of the form

$$(4.29) \quad \text{Res}_{z=0} Q_{Np}(D_{\alpha_1}, \dots, D_{\alpha_n}) \sum_{kq} m^{-z+k-N-1-l} b_N(i\mathcal{P}_{k+1}(m, q + \frac{1}{2})) \bar{\psi}(\frac{q}{\epsilon m}) e^{i\langle q + \frac{1}{2}, \alpha \rangle}.$$

We then observe that the sum is bounded by $\sum_{m=1}^{\infty} m^{-\text{Re}z-N-1+k+n}$, hence converges absolutely and uniformly for $\text{Re}z > -N+k+n$. It follows that for $N > (n+k)$ the sum in (4.29) defines a holomorphic function of z in a half-plane containing $z = 0$ and since the operations of taking the residue in z and derivatives in α commute, each term (4.29) is zero. This justifies (4.28) and shows that it is actually a finite sum in j , say $j < M$ (in fact $M=(k+1)(n+k+1)$).

The residue in (4.28) is therefore well-defined and independent of ϵ . Since $\bar{\psi}(\frac{D_{\alpha}}{m\epsilon})T(\alpha) \rightarrow T(\alpha)$ as $\epsilon \rightarrow \infty$ we must have

$$(4.30) \quad \begin{aligned} a_{\gamma k} &= \text{Res}_{z=0} \sum_{m=1}^{\infty} \sum_{j=0}^M m^{-z+k-j} \mathcal{F}_{k,k-j}(D_{\alpha}, L) T(\alpha) \\ &= \text{Res}_{z=0} \sum_{j=0}^M \zeta(z+j-k) \mathcal{F}_{k,k-j}(D_{\alpha}, L) T(\alpha). \end{aligned}$$

Here, ζ is the Riemann zeta-function, which has only a simple pole at $s = 1$ with residue equal to one. It follows that the only term contributing to (4.29) is that with $j = k + 1$ and hence we have

$$(4.30), \quad a_{\gamma k} = \mathcal{F}_{k,-1}(D_{\alpha}, L) T(\alpha)$$

completing the proof of Theorem C. \square

5. LOCAL FORMULAE FOR THE RESIDUES: PROOF OF THEOREM A

The characterization of the wave invariants in Theorem A is reminiscent of that of the heat invariants in [ABP] or [Gi], but involves non-local metric invariants near γ . We begin by determining the metric data which contribute to $a_{\gamma k}$.

As always, we assume that γ denotes a primitive closed geodesic, and denote the ℓ th iterate of γ by γ^{ℓ} . As above, we use the exponential map along $N_{\gamma} \sim S_L^1 \times \mathbb{R}^n$ to pull back the metric to a metric on a neighborhood of $S_L^1 \times \{0\}$, or more simply S^1 , in $S_L^1 \times \mathbb{R}^n$ with the same wave invariants as g along γ . This reduces the theorem to the case $M = S_L^1 \times \mathbb{R}^n$, with S^1 a closed geodesic of the metric.

We let $J_{S^1}^m$ denote the manifold of m -jets along S^1 of Riemannian metrics on $S^1 \times \mathbb{R}^n$ with the property that S^1 is a closed geodesic. We also let $J_{S^1}^{m_{ell}}$ denote the open submanifold of m -jets of metrics for which S^1 is non-degenerate elliptic. We also write $J_{\gamma_{ell}}^m$ when we wish to identify γ and S^1 . The density $I_{\gamma k}(s)ds$ of the k th wave invariant is then a map

$$I_{\gamma k} : J_{\gamma_{ell}}^{m_k} \rightarrow \Omega^1(S^1)$$

where $\Omega^1(S^1)$ is the space of smooth densities along S^1 and where the jet order m_k will be shown below to be $m_k = 2k + 4$. Since $I_{\gamma k}(s, g)ds$ is independent of the choice of coordinates on $S^1 \times \mathbb{R}^n$ we may express it in terms of Fermi normal coordinates (s, y) with respect to a fixed normal frame $e(s)$ for g . As usual, the metric coefficients $g_{ij}(i, j = 0, 1, \dots, n)$ will be understood relative to these coordinates.

§5.1 The metric data in $I_{k\gamma}$

We now claim that $I_{k\gamma}$ is an invariant polynomial in the following data:

- (i) the curvature tensor R and its covariant derivatives $\nabla^m R$ with $m \leq 2k + 2$, contracted with respect to the Fermi normal vector fields $\frac{\partial}{\partial s}$ and $\sum_{j=1}^n c_j \frac{\partial}{\partial y_j}$ with $c_j \in \mathbb{R}$;
- (ii) the components $\text{Re}y_{ij}(s), \text{Im}y_{ij}(s)$ of the normalized eigenvectors $Y_j \in \mathcal{J}_{\gamma}^{\perp} \otimes \mathbb{C}$ of P_{γ} relative to e_1, \dots, e_n , and their first derivatives;

- (iii) at most $2k + 1$ indefinite integrals over S_L^1 of (i)-(ii).
- (iv) the length L and inverse length L^{-1} of γ^ℓ ;
- (v) the Floquet invariants $\beta_j = (1 - e^{i\alpha_j})^{-1}$;

Indeed, by Theorem C, $I_{\gamma,k}(s, g)ds$ is a density depending only on the data contained in $T(\alpha), L$ and in the coefficients of $\mathcal{F}_{k,-1}$. The latter is identical to the data in the coefficients in $\tilde{p}_1, \dots, \tilde{p}_{k+1}$, hence to that in p_1, \dots, p_{k+1} . By Proposition (4.2), these depends only on $j_\gamma^{2k+4}g$, and on the coefficients of the symplectic matrices $r_\alpha(s), a_s$ which arise in the intertwining of Δ_h to its normal form. The coefficients of r_α depend only on L and the α_j 's and those of a_s depend only on the Jacobi field components y_{ij} , and their first derivatives \dot{y}_{ij} . The second and higher derivatives of y_{ij} can of course be eliminated by means of the Jacobi equation. Regarding (ii), we recall (see [Gr, Theorem 9.21]) that the Taylor series expansions at $y = 0$ of the $g_{ij}(s, y)$'s, of the co-metric coefficients $g^{ij}(s, y)$, and of the volume densities and their powers, involve only the curvature tensor R and its covariant derivatives $\nabla R, \dots, \nabla^m R$ contracted with respect to the normal Fermi vector fields. Since the coefficients $\mathcal{L}_{2-\frac{m}{2}}$ of the semi-classical expansion of Δ_h depend only on this data, and since the intertwining only introduce coefficients in data (ii)-(iv), the $p_k(I)$'s can only involve the g_{ij} 's thru the curvature and its covariant derivatives.

For the fact that $I_{\gamma,k}(s)$ is a polynomial in (i)-(iv), it suffices to reconsider the construction of the normal form. Obviously the coefficients of the \mathcal{L} 's and their metaplectic conjugates the $\mathcal{D}_{\frac{j}{2}}$'s are polynomials in (i)-(iv). It then follows from (2.39c) that the $f_j(I)$'s, hence the $p_j(I)$'s, are also polynomials in this data. Indeed, we argue inductively from the construction of the normal form that the coefficients change in the step from $\mathcal{D}_{\frac{j}{2}}^{\frac{m}{2}}$ to $\mathcal{D}_{\frac{j}{2}}^{\frac{m+1}{2}}$ as a result of taking commutators with operators whose coefficients are polynomials in (i)-(iv). It follows that the coefficients in $\mathcal{D}_{\frac{j}{2}}^{\frac{m+1}{2}}$ are also polynomials in this data, with possibly two more D_s -derivatives due to the commutators with D_s . Since the Jacobi data y_{ij}, \dot{y}_{ij} enters in thru a linear change of variables, the degree of the polynomial in this data will be closely related to the degree in the (y, η) variables, which is twice the degree in the I variables. The degree of the polynomial in the Jacobi data is however not necessarily the same as that (namely, $k+2$) in the I -variables: from the proof of Lemma (2.22), we see that the algorithm for computing the polynomials $f_j(I_1, \dots, I_n)$ involves taking operator commutators (or Poisson brackets of symbols); this lowers the order in the I -variables but not in the Jacobi data which are coefficients of the polynomials in the I 's. We will show below (see 'Jacobi degrees') that the order in the Jacobi data is no more than $6k + 6$.

§5.2 Weights of $I_{k\gamma}$ and of the data

To determine the weights of the polynomials in the data (i)-(iv), we now extend (and in part recall) the table in §1.4 describing how the various data transform under the metric rescaling $g \rightarrow g_\epsilon := \epsilon^2 g$. As above, (s, y) always refer to Fermi normal coordinates relative to g , and for notational simplicity we put $s = y_o$. The notations ∇, R, Δ refer respectively to the Riemannian connection, curvature and Laplacian. We distinguish Δ from the local expression $\frac{1}{\sqrt{g}} \sum_{i,j=o}^n \partial_{y_i} g^{ij} \sqrt{g} \partial_{y_j}$ for Δ relative to the Fermi normal coordinate frame.

| g | $\epsilon^2 g$ |
|-----------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| $g_{ij} = g(\partial_{y_i}, \partial_{y_j})$ | $g_{ij} = \epsilon^2 g(\epsilon^{-1} \partial_{y_i}, \epsilon^{-1} \partial_{y_j})$ |
| $D_{s,y}^\alpha g_{ij}$ | $\epsilon^{- \alpha } D_{s,y}^\alpha g_{ij}$ |
| ∇, R, Δ | $\nabla, R, \epsilon^{-2} \Delta$ |
| $\frac{1}{\sqrt{g}} \sum_{i,j=o}^n \partial_{y_i} g^{ij} \sqrt{g} \partial_{y_j}$ | $\epsilon^{-2} \frac{1}{\sqrt{g}} \sum_{i,j=o}^n \partial_{y_i} g^{ij} \sqrt{g} \partial_{y_j}$ |
| $L, (s, y)$ | $\epsilon L, (\epsilon s, \epsilon y)$ |
| $y_{ij} := g(Y_i, e_j)$ | $\epsilon^{\frac{1}{2}} y_{ij} = \epsilon^2 g(\epsilon^{-\frac{1}{2}} Y_j, \epsilon^{-1} e_j)$ |
| $K_{ij} = g(R(\partial_s, e_i) \partial_s, e_j)$ | $\epsilon^{-2} K_{ij}$ |

The trace of the wave group thus scales as

$$(5.2.1) \quad \text{Tr} e^{it\sqrt{\Delta}} \rightarrow \text{Tr} e^{i\frac{t}{\epsilon}\sqrt{\Delta}}$$

from which it follows that

$$(5.2.2a) \quad a_{\gamma k} = \text{res} \sqrt{\Delta}^k e^{iL\sqrt{\Delta}} \rightarrow \epsilon^{-k} a_{\gamma k}.$$

This can also be seen from the fact that $a_{\gamma k}$ is the coefficient of $(t - L + i0)^k \log(t - L + i0)$ which is homogeneous of degree k modulo smooth functions of t . Since $a_{\gamma k} = \int_{\gamma} I_{\gamma k}(s) ds$, and the integral scales like ds , we also have

$$(5.2.2b) \quad I_{\gamma k}(s, \epsilon^2 g) = \epsilon^{-k+1} I_{\gamma k}(s).$$

As a check on the normal form, let us verify that $\text{wgt}(a_{\gamma k}) = -k$ directly from a weight analysis of the normal form. It is obvious that $\mathcal{R} = \frac{1}{L}(LD_s + H_\alpha)$ scales like ϵ^{-1} and so does each term in the expansion

$$\sqrt{\Delta} \sim \mathcal{R} + \frac{p_1(I_1, \dots, I_n)}{L\mathcal{R}} + \dots$$

Moreover the $p_k(I_1, \dots, I_n)$'s and \mathcal{P}_k 's are also of weight -1 , as determined in Lemma (2.22) and Theorem B. Since the transition from p_ν to the $\tilde{p}_{\nu+r}$'s only involves multiplication of p_ν with the r th power of $\frac{H_\alpha}{LD_s}$ it is obvious that \tilde{p}_ν also has weight -1 . Expanding the exponent in the residue calculation, we get that

$$(5.2.3) \quad a_{\gamma k} = \sum_{N=0}^{k+1} \frac{i^N}{N!} \text{res} \left[\frac{1}{L} (LD_s + H_\alpha) + \mathcal{P}_k \right]^k (L\mathcal{P}_k)^N e^{iH_\alpha}$$

with $[\frac{1}{L}(LD_s + H_\alpha) + \mathcal{P}_k]$ of weight -1 and with $(L\mathcal{P}_k)$ of weight 0 . Hence $\text{wgt}(a_{\gamma k}) = -k$.

§5.3 Wave invariants and QBNF coefficients

They are related as follows: The terms in (5.2.3) contributing nontrivially to $a_{\gamma k}$ have the form

$$(5.3.1a) \quad L^{-k+j_1+\dots+j_{k+1}} \text{res}(LD_s)^\ell \epsilon^{-(j_1+2j_2+\dots+(k+1)j_{k+1})} H_\alpha^{j_o} \tilde{p}_1^{j_1} \tilde{p}_2^{j_2} \dots \tilde{p}_{k+1}^{j_{k+1}} e^{iH_\alpha}$$

with

$$(5.3.1b) \quad j_o + j_1 + \dots + j_{k+1} + \ell = N + k, \quad j_1 + 2j_2 + \dots + (k+1)j_{k+1} = \ell + 1.$$

By the argument in the proof of Theorem C (§4), taking the residue removes the factors of LD_s and replaces an expression $\text{res}(LD)^{-1} F(I) e^{iH_\alpha}$ by the value of $F(D_\alpha + \frac{1}{2}) T(\alpha)$ at a regular point. It follows that

$$(5.3.2) \quad a_{\gamma k} = \sum C_{k;j,\ell} L^{-k+j_1+\dots+j_{k+1}} [H_\alpha^{j_o} \tilde{p}_1^{j_1} \tilde{p}_2^{j_2} \dots \tilde{p}_{k+1}^{j_{k+1}}] (D_\alpha + \frac{1}{2}) T(\alpha)$$

where the sum is taken over the indices specified in (5.3.1b).

§5.4 Jacobi degrees

We now show that the degree of $I_{\gamma k}$ as a polynomial in the Jacobi data y_{ij}, \dot{y}_{ij} is $\leq 6k + 6$. The proof a detailed review of the construction of the polynomials $f_j(I_1, \dots, I_n)$ in Lemma (2.22) as well as the relation between the Jacobi degrees of the polynomials f_j 's and of the operators $\mathcal{F}_{k,-1}$ in Theorem C. For the sake of brevity, and since it is quite routine, we will be a little sketchy in a few of the details. We will use the notation $J - \deg$ for the degree of a polynomial in the metric data above with respect to the Jacobi field components.

(5.4.1) Lemma $J - \deg \tilde{p}_{j+1} = J - \deg p_{j+1} = J - \deg f_j = 6j + 6$.

Proof From the formulae (4.19) relating the $\tilde{p}_j(I_1, \dots, I_n)$'s and the $p_j(I_1, \dots, I_n)$, and from the fact that $\frac{H_\alpha}{D_s}$ has Jacobi degree 0 , we see that

$$(5.4.2a) \quad J - \deg \tilde{p}_j = \max \{J - \deg p_1, \dots, J - \deg p_j\}$$

On the other hand, the relation between the p_j 's and f_j 's is given by comparing (3.3) and (3.12):

$$(5.4.2b) \quad f_j = \frac{2}{L} p_{j+1} + \sum_{i+r+2=j+1} p_{i+1} p_{r+1}$$

Let us assume $J - \deg f_j = 6j + 6$: an easy induction using (5.4.2b) then shows that $J - \deg p_{j+1} = 6j + 6$, and another using (5.4.2a) shows that $J - \deg \tilde{p}_{j+1} = 6j + 6$.

Hence we must prove (5.4.2b). By (2.39c) we have that

$$(5.4.3) \quad J - \deg f_j = J - \deg \mathcal{D}_{-j}^{j+\frac{1}{2}}.$$

To determine the latter, we need to recall that $\mathcal{D}_{-j}^{j+\frac{1}{2}}$ is the h^j -term in the expression

$$Ad(e^{ih^{j+\frac{1}{2}}Q_{j+\frac{1}{2}}})Ad(e^{ih^jQ_j})\dots Ad(e^{ih^{\frac{1}{2}}Q_{\frac{1}{2}}})\mathcal{D}_h$$

with $Ad(V)A := V^{-1}AV$. Using that $Ad(e^{iB}) = e^{iAd(B)}$, and expanding everything in a formal h -series, we find that $\mathcal{D}_{-j}^{j+\frac{1}{2}}$ is the h^j -term in the series

$$(5.4.4) \quad \sum_{n, N_{\frac{1}{2}}, \dots, N_{2j-1}} h^{\frac{n}{2}-2} (h^{j+\frac{1}{2}})^{N_{j+\frac{1}{2}}} (h^j)^{N_j} \dots (h^{\frac{1}{2}})^{N_{\frac{1}{2}}} ad(Q_{j+\frac{1}{2}})^{N_{j+\frac{1}{2}}} \dots ad(Q_{\frac{1}{2}})^{N_{\frac{1}{2}}} \mathcal{D}_{2-\frac{n}{2}}.$$

We claim

$$(5.4.5) \text{ Claim: } \text{Suppose } J - \deg Q_{\frac{m}{2}} = 3m \text{ for } m \leq 2j+1. \text{ Then: } J - \deg \mathcal{D}_{-j}^{j+\frac{1}{2}} = 6j+6.$$

Proof: The h^j term in (5.4.4) is the sum of terms with indices satisfying $\frac{n}{2}-2+N_{j+\frac{1}{2}}(j+\frac{1}{2})+\dots+N_{\frac{1}{2}}(\frac{1}{2})=j$. Multiplying by 6 and using the hypothesis we find that

$$J - \deg \mathcal{D}_{-j}^{j+\frac{1}{2}} = n + 6 \sum_{m=1}^{2j+1} \frac{m}{2} N_{\frac{m}{2}} = n + 6[j+2 - \frac{n}{2}].$$

The claim now follows from the fact that $n \geq 3$ in any term in the sum. □

$$(5.4.6) \text{ Claim: } \text{Suppose } J - \deg \mathcal{D}_{-\frac{j}{2}+\frac{1}{2}}^{\frac{j}{2}} = 3j+3. \text{ Then: } J - \deg Q_{\frac{j+1}{2}} = 3j+3.$$

Proof: Follows from (2.39c,d) which implies that

$$(5.4.7) \quad J - \deg Q_{\frac{j}{2}+\frac{1}{2}} = J - \deg \mathcal{D}_{-\frac{j}{2}+\frac{1}{2}}^{\frac{j}{2}}.$$

□

We then show:

$$(5.4.8) \text{ Claim: } J - \deg Q_{\frac{j}{2}} = 3j.$$

Proof: We prove this by induction on j . For $j=1$ it holds by explicit calculation: from (2.27a), $\deg Q_{\frac{1}{2}} = \deg \mathcal{D}_{\frac{1}{2}} = \deg \mathcal{L}_{\frac{1}{2}} = 3$. It follows then by Claim (5.4.5) (with $j=0$) that $J - \deg \mathcal{D}_0^{\frac{1}{2}} = 6$, and then by Claim (5.4.6) (with $j=1$) that $J - \deg Q_1 = 6$. The rest of the induction proceeds similarly and the details are left to the reader. □

The proof of Lemma (5.4.1) is completed by combining (5.4.3) and Claims (5.4.5) and (5.4.6). □

Finally we have

$$(5.4.9) \text{ Lemma: } J - \deg a_{k\gamma} \leq 6k+6.$$

Proof: As in §5.3, the k th residual terms in $[(D_s + H_\alpha) + \mathcal{P}_k]^k e^{iH_\alpha} e^{iLP_k}$ involve only the products

$$\tilde{p}_1^{j_1} \tilde{p}_2^{j_2} \dots \tilde{p}_{k+1}^{j_{k+1}}$$

of the \tilde{p}_j 's with

$$(5.4.10) \quad j_0 + j_1 + \dots + j_{k+1} + \ell = N + k, \quad j_1 + 2j_2 + \dots + (k+1)j_{k+1} = \ell + 1, \quad \ell \leq k, N \leq k+1.$$

Using that $J - \deg \tilde{p}_m \leq 6m$, the Jacobi degree of any such term cannot exceed $6(j_1 + \dots + j_{k+1})$. We claim that the maximum possible value, subject to the constraints (5.4.10), is $6k+6$. Indeed, since $j_1 + \dots + j_{k+1} = \ell + 1 - (j_2 + 2j_3 + \dots + kj_{k+1}) \leq k+1 - (j_2 + 2j_3 + \dots + kj_{k+1})$ the maximum value is achieved when $j_1 = k+1$ and all other j_m 's are zero. □

This completes the proof of Theorem A. □

6. QUANTUM BIRKHOFF NORMAL FORM COEFFICIENTS

In this section we give a brief summary of the algorithm for calculating the quantum Birkhoff normal form coefficients $B_{\gamma;k,j}$. From these coefficients one could determine the wave invariants $a_{\gamma;k}$ as described in §5.3, but since the $B_{\gamma;k,j}$'s are simpler spectral invariants we choose to concentrate on them. In the next section, we apply the algorithm to the calculation of the coefficients $B_{\gamma;o,j}$ in dimension 2.

Preliminaries

(6.1) Definition *The quantum Birkhoff normal form (QBNF) coefficients are the coefficients of the monomials $I^j := I_1^{j_1} \dots I_n^{j_n} = |z_1|^{2j_1} \dots |z_n|^{2j_n}$ in the complete Weyl symbol $\tilde{p}_k(|z|_1^2, \dots, |z|_n^2)$ of the coefficient operator $\tilde{p}_k(I_1, \dots, I_n)$ of the model normal form of Theorem B:*

$$\tilde{p}_k(|z|_1^2, \dots, |z|_n^2) = \sum_{j \in \mathbb{N}^n : |j| \leq k+1} B_{\gamma;k,j} |z|^{2j}.$$

There is of course something arbitrary in the emphasis on the $\tilde{p}_k(I_1, \dots, I_n)$'s here. The coefficients of the monomials of the complete symbol of the $p_k(I_1, \dots, I_n)$'s are equally spectral invariants and we only prefer the $\tilde{p}_k(I_1, \dots, I_n)$'s to maintain consistency with the terminology of [G.1-2]. Moreover, the coefficients of the operator monomials I^j in either the p_k 's or \tilde{p} 's are also spectral invariants and in view of the relation between wave invariants and the QBNF (§4), it is the operator coefficients which are most simply related to the wave invariants.

The crucial point for the effective calculation of the QBNF coefficients is that they can be obtained from the coefficients of the complete symbol of the semi-classical normal form (SCNF)

$$W_h^* \mathcal{R}_h W_h|_o \sim h^{-2} + \sum_{j=0}^{\infty} h^j f_k(I_1, \dots, I_n)$$

of Lemma (2.22) (after restriction to weight 0.) Indeed, after substituting $h = \mathcal{R}^{-1}$, the square of the QBNF is the SCNF. Hence, the key invariants are really the coefficients of the monomials in the complete symbols

$$f_k(|z_1|^2, \dots, |z_n|^2) = \sum_{j \in \mathbb{N}^n, |j| \leq k+2} c_{\gamma;k,j} |z|^{2j}$$

of the coefficient operators $f_k(I_1, \dots, I_n)$. In the remainder of the section, we drop the subscript γ from the notation.

On the operator level, the relation between the \tilde{p}_k 's, p_k 's and f'_k 's is very simple: the operators $p_k(I_1, \dots, I_n)$ of Theorem B are related to the operators $f_k(I_1, \dots, I_n)$ of Lemma (2.22) by:

$$(6.2a) \quad f_k = \frac{2}{L} p_{k+1} + \sum_{i+j=k, i,j \geq 1} p_i p_j.$$

while the operators \tilde{p}_j are related to the operators p_k by:

$$\tilde{p}_\ell = \sum_{k \geq 1, (j_1, \dots, j_k) : k+|j|=\ell} C_{k,(j_1, \dots, j_k)} H_\alpha^{|j|} p_k$$

for certain universal (multinomial) constants $C_{k,(j_1, \dots, j_k)}$. On the symbol level, the relation is a little more complicated since one has to compose the symbols in the Weyl calculus. In the following we summarise the steps involved in calculating the coefficients of the monomials $|z|^{2j}$ in the complete symbols of the f_k 's. To distinguish on operator from its complete symbol we use the notation $\hat{f}(I_1, \dots, I_n)$ for the Weyl operator with complete symbol $f(|z_1|^2, \dots, |z_n|^2)$. We also recall that the notation $|_o$ refers to the weight 0 part of an operator relative to D_s : $(A_2(s, x, D_x) D_s^2 + A_1(s, x, D_x) D_s + A_0(s, x, D_x))|_o = A_o(s, y, D_y)$. We will assume that the \mathcal{L} 's and \mathcal{D} 's have been expressed in terms of the weightless Fermi normal coordinates (s, x) of §1.5. The complex coordinates z are given by $z = x + i\xi$.

Summary of the algorithm

| Step | Relevant data |
|-----------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Step 1 | Express $\hat{f}_{\frac{N-1}{2}}$ in terms of $\mathcal{D}_{2-\frac{N}{2}}$'s |
| Formula for $\hat{f}_{\frac{N-1}{2}}$ | $\hat{f}_{\frac{N-1}{2}}(I_1, \dots, I_n) := \frac{1}{L} \int_o^L \int_{T^n} V_t^* \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} o V_t dt ds$ (diagonal part; N odd) |
| Equivalently | $\hat{f}_{\frac{N-1}{2}}(q_1 + \frac{1}{2}, \dots, q_n + \frac{1}{2}) := \frac{1}{L} \int_o^L \langle \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} \gamma_q, \gamma_q \rangle ds$ |
| Weyl symbol | $f_{\frac{N-1}{2}}(z_1 ^2, \dots, z_n ^2) = \frac{1}{L} \sum_{ k \leq \frac{N+3}{2}} c_{\frac{N-1}{2};j} z ^{2j}$ |
| Recursion for $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N+1}{2}}$ | $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N+1}{2}} = \text{term of order } h^{\frac{N-1}{2}} \text{ in}$ $\cdot \sum_{n=3}^{N+2} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{N}{2}} h^{\frac{N-1}{2}} \{ [L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} \}$ $\cdot + \sum_{n=N+4}^{\infty} h^{-2+\frac{n}{2}} \mathcal{D}_{2-\frac{n}{2}}^{\frac{N}{2}} + \sum_{m=2}^{\infty} h^{m\frac{N+1}{2}} \frac{i^m}{m!} (ad \tilde{Q}_{\frac{N+1}{2}})^m (L^{-1} D_s)$ $\cdot + \sum_{m=1}^{\infty} \sum_{p=3}^{\infty} h^{m\frac{N+1}{2} + \frac{p}{2} - 2} \frac{i^m}{m!} (ad \tilde{Q}_{\frac{N+1}{2}})^m (\mathcal{D}_{2-\frac{p}{2}}^{\frac{N}{2}})$ |
| Operator Transport : equations for \tilde{Q} 's | $N+1$ odd: $\{ [L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} \} _o = 0$ $N+1$ even: $\{ [L^{-1} D_s, \tilde{Q}_{\frac{N+1}{2}}] + \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} \} _o = f_{\frac{N-1}{2}}(I_1, \dots, I_n)$ |
| Symbol | $N+1$ odd: $L^{-1} \partial_s \tilde{Q}_{\frac{N+1}{2}}(s, z, \bar{z}) = -i \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(s, z, \bar{z})$ |
| Transport equations | $N+1$ even: $L^{-1} \partial_s \tilde{Q}_{\frac{N+1}{2}}(s, z, \bar{z}) = -i (\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(s, z, \bar{z})) + i f_{\frac{N-1}{2}}(z_1 ^2, \dots, z_n ^2)$ |
| Homological equations | $N+1$ odd: $\tilde{Q}_{\frac{N+1}{2}}(0, e^{i\alpha} z, e^{-i\alpha} \bar{z}) - \tilde{Q}_{\frac{N+1}{2}}(0, z, \bar{z}) = -iL \int_0^L \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(s, z, \bar{z}) ds$ $N+1$ even: $\tilde{Q}_{\frac{N+1}{2}}(0, e^{i\alpha} z, e^{-i\alpha} \bar{z}) - \tilde{Q}_{\frac{N+1}{2}}(0, z, \bar{z}) = -iL \int_0^L (\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(s, z, \bar{z})) ds$ $\cdot - f_{\frac{N-1}{2}}(z_1 ^2, \dots, z_n ^2)$ |
| Solutions: | $N+1$ odd: $\tilde{Q}_{\frac{N+1}{2}}(s, z, \bar{z}) = \tilde{Q}_{\frac{N+1}{2}}(0, z, \bar{z}) - iL \int_o^s \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(t, z, \bar{z}) ds$ $N+1$ even: $\tilde{Q}_{\frac{N+1}{2}}(s, z, \bar{z}) = \tilde{Q}_{\frac{N+1}{2}}(0, z, \bar{z})$ $\cdot - iL \int_o^s \{ \mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o(t, z, \bar{z}) - f_{\frac{N-1}{2}}(z_1 ^2, \dots, z_n ^2) \} _o ds$ |
| Step 2 | Express $\mathcal{D}_{2-\frac{N}{2}}$'s in terms of metric data |
| Conjugate to $\mathcal{L}_{2-\frac{N}{2}}$'s | $\mathcal{D}_{2-\frac{N}{2}} = \mu(\mathcal{A}_{\mathcal{L}}^*) \mathcal{L}_{2-\frac{N}{2}} \mu(\mathcal{A}_{\mathcal{L}}^*)^{-1}$ |
| $\mathcal{L}_{2-\frac{N}{2}} =$ $h^{\frac{N}{2}-2}$ -term of | $-(hL)^{-2} g_{[h]}^{oo} + 2i(hL)^{-1} g_{[h]}^{oo} \partial_s + i(hL)^{-1} \Gamma_{[h]}^o$ $+ h^{-1} (\sum_{ij=1}^n g_{[h]}^{ij} \partial_{x_i} \partial_{x_j}) + h^{-\frac{1}{2}} (\sum_{i=1}^n \Gamma_{[h]}^i \partial_{x_i}) + (\sigma)_{[h]}$ |
| Step 3 | Solve for coefficients |
| Coeff's of $\mathcal{D}_{2-\frac{N}{2}}^* _o$ | $\mathcal{D}_{2-\frac{N}{2}}^* _o(s, z, \bar{z}) = \sum_{ m + n \leq 2k} d_{\frac{k}{2}-2; m, n}^*(s) z^m \bar{z}^n$ |
| Coeff's of $\mathcal{D}_{-\frac{N-1}{2}}^{\frac{N}{2}} _o$ | $d_{\frac{N-1}{2}; m, n}^{\frac{N}{2}}(s) = F_{\frac{N-1}{2}; m, n}^{\frac{N}{2}}(d_{\frac{k}{2}-2; m, n}^*; k \leq \frac{N-1}{2})$ |
| Coeff's of $\tilde{Q}_{\frac{N+1}{2}}$'s | $\tilde{Q}_{\frac{N+1}{2}}(s, z, \bar{z}) = \sum_{ m + n \leq N+3} q_{\frac{N+1}{2}; m, n}(s) z^m \bar{z}^n$ |
| Off-diagonal Coefficients | $q_{\frac{N+1}{2}; m, n}(0) = -i(1 - e^{i(m-n)\alpha})^{-1} - iL \int_o^L d_{\frac{N-1}{2}; m, n}^{\frac{N}{2}}(s) ds$ |
| Diagonal Coefficients | $c_{\frac{N-1}{2}; k} = \bar{d}_{\frac{N-1}{2}; k, k}^{\frac{N}{2}} = \frac{1}{L} \int_o^L d_{\frac{N-1}{2}; k, k}^{\frac{N}{2}}(s) ds$ |

Remark Since $\sqrt{\Delta}$ commutes with complex conjugation, the odd order terms in its Weyl symbol are even functions under the involution $(s, \sigma, x, \xi) \rightarrow (s, -\sigma, x, -\xi)$. Although we have restricted to the positive cone where $\sigma > 0$, one gets similar normal forms for $\sigma < 0$ as long as D_s is interpreted as $|D_s|$. Since the Harmonic oscillators and their powers have even symbols, it must be the case that the QBNF coefficients of the even order terms vanish. For instance the coefficient $B_{o;2}$ of $\frac{I}{D_s}$ must vanish automatically.

7. EXPLICIT FORMULAE IN DIMENSION 2

To illustrate the algorithm, we carry out the details of the calculation of $a_{\gamma o}$, or more importantly the normal form coefficients $B_{k;j}$ for $k=0$, in dimension 2. The result may be summarized as follows:

QBNF coefficients for $k=0$, $\dim = 2$ The complete symbol of $f_o(I)$ in complex coordinates $z = y + i\eta$ has the form $B_{o;4}|z|^4 + B_{o;0}$ where $B_{o;j}$ are given for both $j = 0, 4$ by weight -2 Fermi-Jacobi polynomials of the form:

$$\begin{aligned} B_{o;j} = & \frac{1}{L} \int_o^L [a|\dot{Y}|^4 + b_1\tau|\dot{Y}Y|^2 + b_2\tau Re(\bar{Y}\dot{Y})^2 + c\tau^2|Y|^4 + d\tau_{\nu\nu}|Y|^4 + e\delta_{jo}\tau]ds + \\ & + \frac{1}{L} \sum_{0 \leq m, n \leq 3; m+n=3} C_{1;mn} \frac{\sin((n-m)\alpha)}{|(1 - e^{i(m-n)\alpha})|^2} \left| \int_o^L \tau_\nu(s) \bar{Y}^m Y^n(s) ds \right|^2 \\ & + \frac{1}{L} \sum_{0 \leq m, n \leq 3; m+n=3} C_{2;mn} \text{Im} \left\{ \int_o^L \tau_\nu(s) \bar{Y}^m Y^n(s) \left[\int_o^s \tau_\nu(t) \bar{Y}^n Y^m(t) dt \right] ds \right\}. \end{aligned}$$

This corroborates the previous remark that the term B_{ok} vanishes. As will be seen, the corresponding density is a total ∂_s -derivative. Also, the coefficient δ_{jo} of τ is the Kronecker symbol, i.e. equals one if $j = 0$ and vanishes if $j = 4$.

The expressions for the normal form coefficients in higher dimensions are very similar, and it is only for the sake of simplicity that we have stated the result in dimension 2.

The result for the wave coefficient $a_{\gamma o}$ is then very similar, modulo the Floquet factors. Indeed, by (4.30) or by §5.3 the wave invariant is related to the normal form coefficients by

$$a_{\gamma o} = \mathcal{F}_{o,-1}(D_\alpha, L)T(\alpha)$$

with

$$\mathcal{F}_{o,-1}(D_\alpha, L) = L\tilde{p}_1(D_{\alpha_1} + \frac{1}{2}, \dots, D_{\alpha_n} + \frac{1}{2}, L) = Lp_1(D_{\alpha_1} + \frac{1}{2}, \dots, D_{\alpha_n} + \frac{1}{2}).$$

Hence

$$\int_\gamma I_{\gamma o} ds = Lp_1(D_{\alpha_1} + \frac{1}{2}, \dots, D_{\alpha_n} + \frac{1}{2})T(\alpha) = Lp_1(D_{\alpha_1} + \frac{1}{2}, \dots, D_{\alpha_n} + \frac{1}{2})\Pi_{j=1}^n (1 - e^{i\alpha_j})^{-1}$$

In view of (2.33) and the fact that $p_1 = \frac{1}{2}Lf_o$ we have

$$(7.1) \quad a_{\gamma o} = \int_\gamma I_{\gamma o} ds = \frac{1}{2}L^2 f_o(D_{\alpha_1} + \frac{1}{2}, \dots, D_{\alpha_n} + \frac{1}{2})\Pi_{j=1}^n (1 - e^{i\alpha_j})^{-1}.$$

From (7.1) and the formula for the complete symbol of f_o one gets the explicit formula for $a_{\gamma o}$ as stated in the Introduction.

To prove that the $B_{o;j}$ coefficients have the form claimed above, we begin with the expression for f_o from (2.33) or from the table in §6:

$$(7.2) \quad f_o(I_1, \dots, I_n) = \frac{1}{L} \int_o^L \int_{T^n} V_t^* \mathcal{D}_o^{\frac{1}{2}} V_t|_o dt.$$

We also have, from (§2, (2.36) and below), that

$$(7.3) \quad \mathcal{D}_o^{\frac{1}{2}} = \mathcal{D}_o + \frac{1}{2}i[\mathcal{D}_{\frac{1}{2}}, \tilde{Q}_{\frac{1}{2}}].$$

We wish to evaluate the coefficients of the complete symbol of f_o in terms of integrals over γ of Fermi-Jacobi data. At first, we will allow the dimension to be arbitrary; when it is time to substitute in metric expressions we will restrict to dimension 2.

Let us consider first the second term on the right side, which simplifies the expression

$$[\mathcal{D}_{\frac{1}{2}}, \tilde{Q}_{\frac{1}{2}}] + \frac{1}{2}i^2[[L^{-1}D_s, \tilde{Q}_{\frac{1}{2}}], \tilde{Q}_{\frac{1}{2}}]$$

by using $[L^{-1}D_s, \tilde{Q}_{\frac{1}{2}}] = i\mathcal{D}_{\frac{1}{2}}$. We recall from §2 or from the table of §6 that the complete symbol of $\tilde{Q}_{\frac{1}{2}}$ is given (in complex vector notation) by

$$\tilde{Q}_{\frac{1}{2}}(s, z, \bar{z}) = \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z}) + L \int_o^s \mathcal{D}_{\frac{1}{2}}|_o(t, z, \bar{z}) dt, \quad \tilde{Q}_{\frac{1}{2}}(0, e^{i\alpha}z, e^{-i\alpha}\bar{z}) - \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z}) = L \int_o^L \mathcal{D}_{\frac{1}{2}}|_o(t, z, \bar{z}) dt.$$

We note that $\mathcal{D}_{\frac{1}{2}}$ is independent of D_s so that $\mathcal{D}_{\frac{1}{2}}|_o = \mathcal{D}_{\frac{1}{2}}$. It follows that

$$[\mathcal{D}_{\frac{1}{2}}(s), \tilde{Q}_{\frac{1}{2}}(s)] = [\mathcal{D}_{\frac{1}{2}}(s), \tilde{Q}_{\frac{1}{2}}(0)] + [\mathcal{D}_{\frac{1}{2}}(s), \int_o^s \mathcal{D}_{\frac{1}{2}}(t) dt]$$

so that the second term of (7.3) contributes to $f_o(I_1, \dots, I_n)$ the *diagonal part* of

$$(7.4diag) \quad \frac{i}{2L^2} \{ [\tilde{Q}_{\frac{1}{2}}(0, e^{i\alpha}z, e^{-i\alpha}\bar{z}), \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z})] + \frac{1}{2} [\int_o^L \mathcal{D}_{\frac{1}{2}}(s) ds, \int_o^s \mathcal{D}_{\frac{1}{2}}(t) dt] \}.$$

Here, the bracket $[,]$ denotes the commutator of complete symbols in the sense of operator (or complete symbol) composition.

To determine the diagonal part, we write (as usual)

$$\tilde{Q}_{\frac{1}{2}}(0, z, \bar{z}) = \sum_{mn: |m|+|n|=3} q_{\frac{1}{2};mn}(0) z^m \bar{z}^n, \quad \mathcal{D}_{\frac{1}{2}}(s, z, \bar{z}) = \sum_{mn: |m|+|n|=3} d_{\frac{1}{2};mn}(s) z^m \bar{z}^n.$$

Let us denote the operator composition of two complete Weyl symbols a, b by $a \# b$, that is

$$Op^w(a) \circ Op^w(b) = Op^w(a \# b)$$

and recall (cf. [Ho III, Theorem 18.5.4]) that this composition has an asymptotic expansion

$$a \# b \sim ab + iP_1(a, b) + \frac{i^2}{2!} P_2(a, b) + \dots$$

where $P_k(a, b)$ is the higher order Poisson bracket (or transvectant), a bidifferential operator given in complex notation by

$$P_k(a, b) = (\partial_z \partial_{\bar{w}} - \partial_{\bar{z}} \partial_w)^k a(z, \bar{z}) b(w, \bar{w})|_{z=w}.$$

It is well known [loc.cit.] that the commutator is given symbolically by the odd expansion

$$a \# b - b \# a \sim \frac{1}{i} P_1(a, b) + \frac{1}{i^3 3!} P_3(a, b) + \dots$$

while anticommutators involve only the even transvectants. Since the complete symbols in (7.4 diag) are homogeneous polynomials of degree 3, the commutators involve only P_1 and P_3 . One easily computes that

$$P_1(z^m \bar{z}^n, z^\mu \bar{z}^\nu) = C_{1;mn\mu\nu} z^{m+\mu-1} \bar{z}^{n+\nu-1}$$

where $C_{1;mn\mu\nu} = \frac{1}{2} \sigma((m, n), (\mu, \nu))$ with σ the standard symplectic inner product, and that

$$P_3(z^m \bar{z}^n, z^\mu \bar{z}^\nu) = C_{(m,n),(\mu,\nu)} z^{m+\mu-3} \bar{z}^{n+\nu-3}$$

for certain other coefficients $C_{(m,n),(\mu,\nu)}$. A straightforward computation then shows that the diagonal part in (7.4 diag) is the sum of the following terms:

$$(7.5.1) \quad \frac{i}{2L^2} \{ [\tilde{Q}_{\frac{1}{2}}(0, e^{i\alpha}z, e^{-i\alpha}\bar{z}), \tilde{Q}_{\frac{1}{2}}(0, z, \bar{z})] \\ = L^{-2} [C_{1;3030} q_{\frac{1}{2};30}(0) q_{\frac{1}{2};03}(0) e^{i3\alpha} + C_{1;2112} q_{\frac{1}{2};21}(0) q_{\frac{1}{2};12}(0) e^{i2\alpha} + C_{1;1221} q_{\frac{1}{2};12}(0) q_{\frac{1}{2};21}(0) e^{i\alpha} + C_{1;0330} q_{\frac{1}{2};03}(0) q_{\frac{1}{2};30}(0)] |z|^4 + \\ + [C_{3;3030} q_{\frac{1}{2};30}(0) q_{\frac{1}{2};03}(0) e^{i3\alpha} + C_{3;2112} q_{\frac{1}{2};21}(0) q_{\frac{1}{2};12}(0) e^{i2\alpha} + C_{3;1221} q_{\frac{1}{2};12}(0) q_{\frac{1}{2};21}(0) e^{i\alpha} + C_{3;0330} q_{\frac{1}{2};03}(0) q_{\frac{1}{2};30}(0)] \\ \text{plus}$$

$$(7.5.2) \quad \frac{1}{2} \int_o^L \int_o^s [\tilde{D}_{\frac{1}{2}}(s, z, \bar{z}), \tilde{D}_{\frac{1}{2}}(t, z, \bar{z})] ds dt \\ = \int_o^L \int_o^s [C_{1;3030} d_{\frac{1}{2};30}(s) d_{\frac{1}{2};03}(t) + C_{1;2112} d_{\frac{1}{2};21}(s) d_{\frac{1}{2};12}(t) + C_{1;1221} d_{\frac{1}{2};12}(s) d_{\frac{1}{2};21}(t) + C_{1;0330} d_{\frac{1}{2};03}(s) d_{\frac{1}{2};30}(t)] ds dt + \\ \int_o^L \int_o^s [C_{3;3030} d_{\frac{1}{2};30}(s) d_{\frac{1}{2};03}(t) + C_{3;2112} d_{\frac{1}{2};21}(s) d_{\frac{1}{2};12}(t) + C_{3;1221} d_{\frac{1}{2};12}(s) d_{\frac{1}{2};21}(t) + C_{3;0330} d_{\frac{1}{2};03}(s) d_{\frac{1}{2};30}(t)] ds dt.$$

We observe that there is no term of order $|z|^2$. Using that

$$q_{\frac{1}{2};mn} = -iL(e^{i\alpha \cdot (m-n)} - 1)^{-1} \int_o^L d_{\frac{1}{2};mn}(s) ds$$

we reduce our problem to the calculation of the $d_{\frac{1}{2};mn}(t)$'s and the $d_{o;mn}(t)$'s.

To evaluate the expressions $\mathcal{D}_{\frac{1}{2}}$ and \mathcal{D}_o . we conjugate back to the \mathcal{L} 's:

$$\mathcal{D}_o = \mu(\mathcal{A}_L^*)\mathcal{L}_o\mu((\mathcal{A}_L^*)^{-1}), \quad \mathcal{D}_{\frac{1}{2}} = \mu(\mathcal{A}_L^*)\mathcal{L}_{\frac{1}{2}}\mu(\mathcal{A}_L^*)^{-1} \quad (7.6a)$$

with

$$(7.6b) \quad \mathcal{A}_L := \begin{pmatrix} L^{\frac{1}{2}}Im\dot{Y} & L^{\frac{1}{2}}Re\dot{Y} \\ L^{-\frac{1}{2}}ImY & L^{-\frac{1}{2}}ReY \end{pmatrix}.$$

Using (2.11) we then compute the \mathcal{L} 's in (unscaled) Fermi normal coordinates as follows:

$$(7.7.1/2) \quad \mathcal{L}_{\frac{1}{2}} = - \sum_{\beta:|\beta|=3} \frac{1}{3!} \partial_y^\beta g^{oo}(s,0) y^\beta$$

$$(7.7.1) \quad \mathcal{L}_o = - \sum_{\beta:|\beta|=4} \frac{1}{4!} \partial_y^\beta g^{oo}(s,0) y^\beta - \sum_{\beta:|\beta|=2} \partial_x^\beta [g^{oo} D_s + J^{-\frac{1}{2}} D_s (g^{oo} J^{\frac{1}{2}})]|_{y=0} y^\beta$$

$$+ \sum_{\beta:|\beta|=2} \sum_{ij=1}^n \partial_y^\beta g^{ij}(s,0) y^\beta \partial_{y_i} \partial_{y_j} + \sum_{i,j=1}^n \partial_{y_j} \Gamma^i(s,0) y_j \partial_{y_i}$$

$$+ \partial_s^2 + C_n \tau(s,0)$$

where τ is the scalar curvature. The last term comes from $\Delta_{\frac{1}{2}} 1$.

We then change variables $y = Lx$ and conjugate the symbols. By metaplectic covariance of the Weyl calculus, the conjugations change the complete Weyl symbols of the \mathcal{L} 's (in the x variables) by the linear symplectic transformation \mathcal{A}_L , i.e by the substitutions

$$(7.8) \quad \begin{aligned} x &\rightarrow L^{-\frac{1}{2}} [(ReY)x + (ImY)\xi] = \frac{1}{2} L^{-\frac{1}{2}} [\bar{Y} \cdot z + Y \cdot \bar{z}] \\ \xi &\rightarrow L^{\frac{1}{2}} [Re\dot{Y}x + (Im\dot{Y})\xi] = \frac{1}{2} L^{\frac{1}{2}} [\bar{Y} \cdot z + \dot{Y} \cdot \bar{z}] \end{aligned}$$

It is evident that the normal form coefficients are going to be rather lengthy. To give the flavor of the full calculations in the simplest setting, we now specialize to the case of surfaces. Later we will briefly extend the calculations to all dimensions.

Dimension 2

In dimension 2 we have (in scaled Fermi coordinates)

$$g^{oo}(s, y) = 1 + C_1 \tau(s) y^2 + C_2 \tau_\nu(s) y^3 + \dots \quad g^{11} = 1 \quad J(s, u) = \sqrt{g^{oo}} = 1 + C'_1 \tau(s) y^2 + \dots$$

, for some constants C_j, C'_j which will change from line to line. Hence the 1/2-density Laplacian in Fermi normal coordinates equals

$$\begin{aligned} -\Delta &= J^{-1/2} \partial_s g^{oo} J \partial_s J^{-1/2} + J^{-1/2} \partial_y J \partial_y J^{-1/2} \\ &\equiv g^{oo} \partial_s^2 + \Gamma^o \partial_s + \partial_y^2 + \Gamma^1 \partial_y + \sigma_o \end{aligned}$$

and the rescaled Laplacian equals

$$-\Delta_h = -(Lh)^{-2} g_{[h]}^{oo} + 2i(Lh)^{-1} g_{[h]}^{oo} \partial_s + i(Lh)^{-1} \Gamma_{[h]}^o - g_{[h]}^{oo} \partial_s^2 + \Gamma_{[h]}^o \partial_s + \partial_y^2 + h^{-\frac{1}{2}} \Gamma_{[h]}^1 \partial_y + (\sigma)_{[h]}$$

Using the Taylor expansion of the metric coefficients one finds that the \mathcal{L} 's have the form

$$(7.9) \quad \mathcal{L}_{\frac{1}{2}} = CL^{-2} \tau_\nu(s) y^3, \quad \mathcal{L}_o = C_1 L^{-2} y^4 \tau_{\nu\nu} + C_2 L^{-1} y^2 \tau \partial_s + C_3 L^{-1} \tau_s y^2 - \partial_s^2 + C_4 \tau y \partial_y + C_5 \tau.$$

All terms have weight -2.

We now switch to the weightless coordinates $y = Lx$ and get:

$$(7.9) \quad \mathcal{L}_{\frac{1}{2}} = CL \tau_\nu(s) x^3, \quad \mathcal{L}_o = C_1 L^2 x^4 \tau_{\nu\nu} + C_2 L^1 x^2 \tau \partial_s + C_3 L \tau_s x^2 - \partial_s^2 + C_4 \tau x \partial_x + C_5 \tau.$$

Making the linear symplectic substitutions above we first get

$$\mathcal{D}_{\frac{1}{2}}(s, z, \bar{z}) = CL^{-\frac{1}{2}} \tau_\nu(s) ([\bar{Y} \cdot z + Y \cdot \bar{z}])^3$$

hence

$$(7.10.1) \quad d_{\frac{1}{2};mn}(s) = C_{mn;3} L^{-\frac{1}{2}} \tau_\nu [\bar{Y}^m \cdot Y^n](s)$$

$$(7.10.2) \quad \bar{d}_{\frac{1}{2};mn}(s) = C_{mn;3} L^{-\frac{3}{2}} \int_o^L \tau_\nu [\bar{Y}^m \cdot Y^n](s) ds$$

$$(7.10.3) \quad q_{\frac{1}{2};mn} = -i C_{mn;3} (1 - e^{i(m-n)\alpha})^{-1} L^{\frac{1}{2}} \int_o^L \tau_\nu [\bar{Y}^m \cdot Y^n](s) ds$$

with $m+n=3$ and for certain coefficients $C_{mn;\beta}$. It is evident that $d_{mn}(s)$ is a Fermi-Jacobi polynomial of weight -2 and of Jacobi degree 3, so that the terms (7.5.1-2) are of Jacobi degree 6 as stated in Theorem A. The diagonal part has terms of degree $|z|^4$ and $|z|^o$ with coefficients of the form (with $m, n = 0, \dots, 3; m+n=3$):

$$(7.11.1) \quad \frac{1}{L} \left\{ \left[\int_o^L \tau_\nu(s) \bar{Y}^m Y^n(s) ds \right] \left[\int_o^L \tau_\nu(t) \bar{Y}^n Y^m(t) dt \right] \frac{e^{i(n-m)\alpha}}{|1 - e^{i(n-m)\alpha}|^2} \right. \\ \left. - \left[\int_o^L \tau_\nu(s) \bar{Y}^n Y^m(s) ds \right] \left[\int_o^L \tau_\nu(t) \bar{Y}^m Y^n(t) dt \right] \frac{e^{-i(m-n)}}{|1 - e^{-i(m-n)}|^2} \right\}$$

and

$$(7.11.2) \quad \frac{1}{L} \left\{ \int_o^L \tau_\nu(s) \bar{Y}^m Y^n(s) \left[\int_o^s \tau_\nu(t) \bar{Y}^n Y^m(t) dt \right] ds - \int_o^L \tau_\nu(s) \bar{Y}^n Y^m(s) \left[\int_o^s \tau_\nu(t) \bar{Y}^m Y^n(t) dt \right] ds \right\}.$$

To calculate the complete symbol of D_s -weight 0, $\mathcal{D}_o|_o$, of the second term $\mathcal{D}_o^{\frac{1}{2}}$ we make the same linear substitution and eliminate any D_s appearing all the way to the right. We also invert the relation

$$\mu(\mathcal{A}_L^*)^{-1} D_s \mu(\mathcal{A}_L^*) = D_s - \frac{1}{2} (L^{-1} \partial_x^2 + L \tau x^2)$$

to get

$$\mu(\mathcal{A}^*) D_s \mu(\mathcal{A}^*)^{-1} = D_s - \frac{1}{2} \mu(\mathcal{A}^*) (L^{-1} \partial_x^2 + L \tau x^2) \mu(\mathcal{A}^*)^{-1}$$

and transform the complete symbol of quadratic term by the symplectic substitution. The result is that $\mathcal{D}_o|_o(s, z, \bar{z})$ equals

$$(7.12.1) \quad C_1 \tau_{\nu\nu} [\bar{Y}z + Y\bar{z}]^4$$

$$(7.12.2) \quad + C_2 L^{-2} \tau [\bar{Y}z + Y\bar{z}]^2 \# ([\bar{Y}z + \dot{Y}\bar{z}]^2 + \tau L^{-2} [\bar{Y}z + Y\bar{z}]^2)$$

$$(7.12.3) \quad + C_3 \tau_s [\bar{Y}z + Y\bar{z}]^2$$

$$(7.12.4.1) \quad - 2L^{-2} \partial_s ([\bar{Y}z + \dot{Y}\bar{z}]^2 - L^{-2} \tau [\bar{Y}z + Y\bar{z}]^2)$$

$$(7.12.4.2) \quad + L^{-2} \{ [\bar{Y}z + \dot{Y}\bar{z}]^2 - L^{-2} \tau [\bar{Y} \cdot z + Y\bar{z}]^2 \} \# \{ [\bar{Y}z + \dot{Y}\bar{z}]^2 - L^{-2} \tau [\bar{Y}z + Y\bar{z}]^2 \}$$

$$(7.12.5) \quad + C_4 L^{-1} \tau (\bar{Y}z + Y\bar{z}) \# (\bar{Y}z + \dot{Y}\bar{z}) + C_5 \tau.$$

Our concern is with the diagonal part of the complete symbol, that is, with the terms involving $|z|^4, |z|^2, |z|^o$, and more precisely with their integrals over γ . To begin with, we observe that the diagonal part of term (7.12.1) is purely of degree $|z|^4$ and its average over γ equals

$$(7.13.1) \quad (Const.) |z|^4 \cdot \frac{1}{L} \int_o^L \tau_{\nu\nu} |Y|^4 ds.$$

The diagonal part of term (17.12.2) contributes only the P_o and P_2 terms, of degrees $|z|^4$ and $|z|^o$ respectively, whose averages over γ have the form

$$(7.13.2) \quad (|z|^4 / or / |z|^o) \cdot \frac{1}{L} \int_o^L \tau [a\tau |Y|^4 + b \operatorname{Re}(\bar{Y}\dot{Y})^2 + c |Y\dot{Y}|^2] ds$$

with constants a, b, c, d which can differ between the two degrees. The missing P_1 -term vanishes: it is a multiple of the Poisson bracket

$$P_1([\bar{Y}z + Y\bar{z}]^2, \tau[\bar{Y}z + Y\bar{z}]^2)$$

which simplifies to a term of the form

$$\tau[\bar{Y}^2\dot{Y}^2 - Y^2\dot{\bar{Y}}^2] = \tau(\bar{Y}\dot{Y} - Y\dot{\bar{Y}})(\bar{Y}\dot{Y} + Y\dot{\bar{Y}}) = C\tau\frac{d}{ds}|Y|^2$$

by the symplectic normalization of the Jacobi eigenfield. However the integral over γ of this term vanishes, that is

$$(7.13.3) \quad \int_o^L \tau_s |Y|^2 = 0.$$

This can be seen from the Jacobi equation, which implies:

$$[\bar{Y}(Y')'' + \tau_s |Y|^2 + \tau Y' \bar{Y}] = 0;$$

integrating over γ and integrating the first term by parts twice kills the outer terms and hence the inner one. In a similar way, the diagonal part of (7.12.3) is of pure degree $|z|^2$ with coefficient $\tau_s |Y|^2$, so it makes no contribution either. Nor does the term (7.12.4.1), which is manifestly a total derivative and hence automatically has zero integral. The term (7.12.4.2) is a composition square and hence contributes only a product P_o -term of degree $|z|^4$ and a P_2 -term of degree $|z|^o$, namely (for $j=0,2$) the diagonal part of

$$P_j(z^2\dot{\bar{Y}}^2 + 2|z|^2|\dot{Y}|^2 + \bar{z}^2\dot{Y}^2 + \tau(z^2\bar{Y}^2 + 2|z|^2|Y|^2 + \bar{z}^2Y^2, z^2\dot{\bar{Y}}^2 + 2|z|^2|\dot{Y}|^2 + \bar{z}^2\dot{Y}^2 + \tau(z^2\bar{Y}^2 + 2|z|^2|Y|^2 + \bar{z}^2Y^2))$$

whose average over γ has the form

$$(7.13.4.2) \quad (|z|^4/or/|z|^o) \cdot \frac{1}{L} \int_o^L [a|\dot{Y}|^4 + b\tau Re(\dot{Y}^2 Y^2) + c\tau |\dot{Y}Y|^2 + d\tau^2 |Y|^4] ds$$

where again the coefficients may vary between the two degrees. Finally, we the first term of (7.12.5) obviously has no diagonal part while obviously the second term contributes

$$(7.13.5) \quad C \frac{1}{L} \int_o^L \tau ds.$$

This completes the analysis of the QBNF coefficients B_{o4}, B_{o2}, B_{o0} .

8. APPENDIX : THE CLASSICAL BIRKHOFF NORMAL FORM

The method of §2 for putting $\sqrt{\Delta}$ into quantum Birkhoff normal form began, essentially, by putting the linearization of $\sqrt{\Delta}$ at γ into normal form by a linear symplectic transformation, and then proceeded by induction on the jet filtration at γ . The purpose of this appendix is to describe, rather briefly and sketchily, how to put the principal symbol of $\sqrt{\Delta}$ into Birkhoff normal form by an analogous method. (We have not found this particular algorithm in the literature, but it is quite likely that it, or a much better algorithm, is well-known). We hope that the classical algorithm will help clarify the procedure in the quantum case.

In the usual Fermi symplectic normal coordinates (s, σ, y, η) , we may write the principal symbol of $\sqrt{\Delta}$ in the form

$$\begin{aligned} \tilde{H}(s, \sigma, y, \eta) &:= (g^{oo}(s, y)\sigma^2 + \sum_{ij=1}^n g^{ij}(s, y)\eta_i\eta_j)^{\frac{1}{2}} \\ &= \sigma(g^{oo}(s, y) + \sum_{ij=1}^n \frac{\eta_i\eta_j}{\sigma^2})^{\frac{1}{2}}. \end{aligned}$$

Taking the Taylor expansion at $y = \eta = 0$ we get

$$\tilde{H}(s, \sigma, y, \eta) = \sigma(1 + \frac{1}{2}[\sum_{ij=1}^n K_{ij}(s)y_i y_j + \sum_{i=1}^n \frac{\eta_i^2}{\sigma^2}] + \dots)$$

from which we extract the *linearized symbol*

$$(A.1) \quad \tilde{h}(s, \sigma, y, \eta) = \sigma + \frac{1}{2} \sum_{ij=1}^n K_{ij}(s)y_i y_j \sigma + \sum_{i=1}^n \frac{\eta_i^2}{\sigma}.$$

We make the symplectic change of variables

$$(A.2) \quad \phi : (s, \sigma, y, \eta) \rightarrow (s', \sigma', y', \eta') := (s + \frac{1}{2} \frac{y\eta}{\sigma}, \sqrt{\sigma}y, \frac{\eta}{\sqrt{\sigma}})$$

which transforms \tilde{h} into

$$(A.3) \quad h(s, \sigma, y, \eta) = \sigma + \frac{1}{2} \sum_{ij=1}^n K_{ij}(s) y_i y_j + \sum_{i=1}^n \eta_i^2$$

and \tilde{H} into

$$H(s, \sigma, y, \eta) = h + h^{[3]} + \dots$$

with \dots denoting terms of order 3 and higher in (y, η) . (Such terms are of order 3/2 with respect to the order in the isotropic calculus, while h is of order 1, which is the rationale for calling h the linearized symbol).

The first step in putting H into Birkhoff normal form is to put h into Birkhoff normal form

$$\hat{h} = \sigma + \frac{1}{2} \sum_{i=1}^n \alpha_i (y_i^2 + \eta_i^2)$$

by means of a symplectic map. Equivalently, we wish to convert the Hamilton equations

$$\begin{aligned} \frac{d}{ds} s &= 1 \\ \frac{d}{ds} \sigma &= 0 \\ \frac{d}{ds} y &= \eta \\ \frac{d}{ds} \eta &= -K(s)y \end{aligned}$$

into the linear equations

$$\begin{aligned} \frac{d}{ds} s &= 1 \\ \frac{d}{ds} \sigma &= 0 \\ \frac{d}{ds} q &= \alpha p \\ \frac{d}{ds} p &= -\alpha q. \end{aligned}$$

We first do this in just the (y, η, q, p) variables, with a symplectic map of the form $(y, \eta) \rightarrow (q, p) = B(s)(y, \eta)$. The condition on B is that

$$(A.4) \quad \dot{B}B^{-1} + B\tilde{K}B^{-1} = \alpha\dot{J}$$

where \tilde{K} is the block anti-diagonal matrix with blocks I and $-K$, where J is the usual block anti-diagonal matrix with blocks $\pm I$ and where $\alpha\dot{J}$ is the block anti-diagonal matrix with coefficients $\pm\alpha_j$ replacing ± 1 in J . The equation (A.4) is equivalent to

$$(A.5) \quad -\frac{d}{ds} B^{-1} + \tilde{K}(s)B^{-1}\alpha\dot{B}^{-1}J$$

which has the solution

$$(A.6) \quad B(s) = r_\alpha(s)a_s^{-1}.$$

To make the map symplectic in all the (s, σ, y, η) variables, we observe that the map

$$\psi_1 : (s, \sigma, y, \eta) \rightarrow (s, \sigma - \frac{1}{2} \sum_{i=1}^n \alpha_i (y_i^2 + \eta_i^2), r_\alpha(s)(y, \eta))$$

is symplectic with respect to $ds \wedge d\sigma + dy \wedge d\eta$ and satisfies

$$\psi_1^*(\sigma + \frac{1}{2} \sum_{i=1}^n \alpha_i (y_i^2 + \eta_i^2)) = \sigma.$$

Also, the map

$$\psi_2 : (s, \sigma, y, \eta) \rightarrow (s, \sigma + f(s, y, \eta), a_s^{-1}(y, \eta))$$

with $f(s, y, \eta) := -\frac{1}{2} \sum_{ij=1}^n K_{ij}(s) y_i y_j + \sum_{i=1}^n \eta_i^2$ is symplectic with respect to $ds \wedge d\sigma + dy \wedge d\eta$ and satisfies

$$\psi_2^*(\sigma) = \sigma + f.$$

It follows that $\chi_1 := \psi_2 \cdot \psi_1$ pulls back h to its Birkhoff normal form.

Let us now write

$$H_1 := \chi_1^*(H) = \hat{h} + h_1^{[3]} + h_1^{[4]} + \dots$$

with $h_1^{[k]}(s, \sigma, y, \eta)$ of order k and vanishing to order k at $(y, \eta) = (0, 0)$. Following the algorithm for putting a Hamiltonian with non-degenerate minimum at 0 into Birkhoff normal form [AM, 5.6.8, p.500] we seek a symplectic map of the form

$$\chi_2 = \exp \sigma^{\frac{1}{2}} F_3$$

with $F_3 = F_3(s, y, \eta)$ vanishing to order 3 and with

$$(A.7) \quad \{\hat{h}, [\chi_2^* H_1]^{\leq 3}\} = 0.$$

Here, $\{, \}$ denotes the Poisson bracket with respect to $ds \wedge d\sigma + dy \wedge d\eta$, $[F]^{\leq k}$ denotes the terms of vanishing order $\leq k$ and $adF \cdot G := \{F, G\}$. The equation (A.7) is equivalent to

$$(A.8) \quad h_1^{[3]} + \{\sigma^{\frac{1}{2}} F_3, \hat{h}\} \in \ker(ad\hat{h}) \cap \mathcal{P}^3$$

where \mathcal{P}^k denotes the space of homogeneous polynomials of degree k in (y, η) with smooth coefficients in s and with an overall factor of $\sigma^{\frac{k}{2}-1}$. In terms of the complex coordinates $z_j = y_j + i\eta_j$ on \mathbb{R}^{2n} we may write

$$F_3 = \sum_{|j|+|k|=3} c_{jk}(s) z^j \bar{z}^k$$

$$h_1^{[3]} = \sum_{|j|+|k|=3} a_{jk}(s) z^j \bar{z}^k$$

with $j = (j_1, \dots, j_n), k = (k_1, \dots, k_n)$, and we may write the Lie derivative \mathcal{L} with respect to the Hamilton vector field of \hat{h} as

$$\mathcal{L} = \frac{\partial}{\partial s} + \sum_{i=1}^n \alpha_i (z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i})$$

The monomials in $\ker ad\mathcal{L} \cap \mathcal{P}^k$ are of the form $z^j \bar{z}^k$ with $\langle \alpha, j - k \rangle = 0$ which implies $j = k$ with our assumptions on α . No such terms occur for odd k , so equation (A.8) thus becomes

$$(A.9) \quad \sum_{|j|+|k|=3} \dot{c}_{jk}(s) + \alpha \cdot (j - k) c_{jk}(s) = -a_{jk}(s).$$

Expanding the periodic (or more generally almost periodic) functions c_{jk} and a_{jk} in Fourier series

$$c_{jk}(s) = \sum e^{ims} \hat{c}_{jk}(m) \quad a_{jk}(s) = \sum e^{ims} \hat{a}_{jk}(m)$$

we can solve (A.9) with

$$(A.10) \quad \hat{c}_{jk}(m) = \frac{\hat{a}_{jk}(m)}{im + \langle \alpha, j - k \rangle}.$$

Writing $H_2 = \chi_2^* \cdot \chi_1^* H$, we arrive at the analogous problem for the fourth order terms. As in the quantum case, the even steps behave a little differently from the odd ones since now there can be terms $H_2^{[4]o}$ in $H_2^{[4]}$ with $|j| = |k|$ and hence which lie in $\ker ad\mathcal{L} \cap \mathcal{P}^4$. Since these terms already commute with \mathcal{L} it suffices to solve the analogue of (A9) with only the coefficients $a_{jk}(s)$ coming from $H_2^{[4]} - H_2^{[4]o}$.

This puts the terms up to fourth order in normal form, and the process continues inductively to define symplectic maps $\chi_N \cdot \chi_{N-1} \cdots \chi_1$ which pulls back H to a normal form up to degree N . Since $\chi_N = I \text{ mod } O_N$ the infinite product defines a smooth symplectic map which pulls back H to a normal form modulo O_∞ , that is, to its Birkhoff normal form.

9. INDEX OF NOTATION

In the following, τ_L will denote the translation operator $\tau_L f(s, y) = f(s + L, y)$ on functions on $\mathbb{R} \times \mathbb{R}^n$. Operators A then transform under τ_L by $\tau_L A \tau_L^*$.

NI.1: Model objects

| Model object | \cdot |
|-------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|
| Functions | $\tau_L f = f$ |
| Operators | $\tau_L A \tau_L^* = A$ |
| Maximal abelian algebra | $\mathcal{A} = \langle \mathcal{R}, I_1, \dots, I_n \rangle$ |
| Distinguished element | $\mathcal{R} := \frac{1}{L}(LD_s + H_\alpha)$ |
| Harmonic Oscillator | $H_\alpha := \frac{1}{2} \sum_{k=1}^n \alpha_k I_k$ |
| Gaussians/Hermite | $\gamma_o(y) = \gamma_{iI}(q) = e^{-\frac{1}{2} y ^2} \quad \gamma_q = C_q A_1^{*q_1} \dots A_n^{*q_n}$ |
| \mathcal{A} -Eigenfunctions | $\phi_{kq}^o(s, y) := e_k(s) \otimes \gamma_q(y)$ |
| \mathcal{R} -Eigenvalues | $\mathcal{R} \phi_{kq}^o(s, y) = r_{kq} \phi_{kq}^o(s, y), \quad r_{kq} = \frac{1}{L}(2\pi k + \sum_{j=1}^n (q_j + \frac{1}{2}) \alpha_j)$ |
| Scaled Laplacian | $\mathcal{R}_h := \mu(\tilde{a}) \Delta_h \mu(\tilde{a})^* \sim \sum_{m=0}^\infty h^{(-2+\frac{m}{2})} \mathcal{R}_{2-\frac{m}{2}}$ |
| Intertwiner to SC normal form | $W_h := \prod_{k=1}^\infty W_{h\frac{k}{2}} \mu(r_\alpha), \quad W_{h\frac{k}{2}} := \exp(ih\frac{k}{2} Q_{\frac{k}{2}})$ |
| S.C. Normal form | $W_h^* \mathcal{R}_h W_h _o \sim h^{-2} + \sum_{j=0}^\infty h^j f_j(I_1, \dots, I_n)$ |

NI.2: Twisted model objects

| Twisted model object | \cdot |
|-------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|
| Functions | $\tau_L f = \mu(r_\alpha(L)) f$ |
| Operators | $\tau_L A \tau_L^* = \mu(r_\alpha(L)) A \mu(r_\alpha(L))^*$ |
| Maximal abelian algebra | $\mathcal{A} = \langle D_s, I_1, \dots, I_n \rangle$ |
| Distinguished element | D_s |
| Harmonic Oscillator | $H_\alpha := \frac{1}{2} \sum_{k=1}^n \alpha_k I_k$ |
| Gaussians/Hermite | $\gamma_o(y) = \gamma_{iI}(q) = e^{-\frac{1}{2} y ^2} \quad \gamma_q = C_q A_1^{*q_1} \dots A_n^{*q_n}$ |
| \mathcal{A} -Eigenfunctions | $e^{ir_{kq}s} \otimes \gamma_q(y)$ |
| D_s -Eigenvalues | $D_s e^{ir_{kq}s} \otimes \gamma_q(y) = r_{kq} e^{ir_{kq}s} \otimes \gamma_q(y)$ |
| Scaled Laplacian | $\mathcal{D}_h := \mu(a) \Delta_h \mu(a)^* \sim \sum_{m=0}^\infty h^{(-2+\frac{m}{2})} \mathcal{D}_{2-\frac{m}{2}}$ |
| Intertwiner to SC normal form | $\tilde{W}_h := \prod_{k=1}^\infty \tilde{W}_{h\frac{k}{2}} \mu(r_\alpha), \quad \tilde{W}_{h\frac{k}{2}} := \exp(ih\frac{k}{2} \tilde{Q}_{\frac{k}{2}})$ |
| S.C. Normal form | $\tilde{W}_h^* \mathcal{D}_h \tilde{W}_h _o \sim h^{-2} + \sum_{j=0}^\infty h^j f_j(I_1, \dots, I_n)$ |

NI.3: Adapted model objects

| Adapted model object | \cdot |
|-------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Functions | $\tau_L f = \mu(T) f$ |
| Operators | $\tau_L A \tau_L^* = \mu(T) A \mu(T)^*$ |
| Maximal abelian algebra | $\mathcal{A} = \langle \mathcal{L}, \Lambda_1 \Lambda_1^*, \dots, \Lambda_n \Lambda_n^* \rangle$ |
| Distinguished element | $\mathcal{L} = D_s - \frac{1}{2} (\sum_{j=1}^n D_{u_j}^2 + \sum_{i,j=1}^n K_{ij}(s) u_i u_j)$ |
| Harmonic Oscillator | $H_\alpha := \frac{1}{2} \sum_{k=1}^n \alpha_k \Lambda_k \Lambda_k^*$ |
| Gaussians/Hermite | $\mu(a^{-1}) \gamma_o(s, u) := U_o(s, u) = (\det Y(s))^{-1/2} \exp(\frac{1}{2} \langle \Gamma(s) u, u \rangle), \quad \Gamma(s) := \frac{dY}{ds} Y^{-1};$ $\mu(a^{-1}) \gamma_q := U_q = C_q \Lambda_1^{q_1} \dots \Lambda_n^{q_n} U_o.$ |
| \mathcal{A} -Eigenfunctions | $\mu(\tilde{a}_s)(\phi_{kq}^o) = \phi_{kq} := e^{ir_{kq}s} U_q(s, u)$ |
| Scaled Laplacian | $\Delta_h \sim \sum_{m=0}^\infty h^{(-2+m/2)} \mathcal{L}_{2-m/2}$ |

NI.4 Intertwining operators

1. $\mu(r_\alpha) := \int_{S^1}^\oplus \mu(r_\alpha(s)) ds$
2. $a_s := \begin{pmatrix} Im \dot{Y}(s)^* & Im Y(s)^* \\ Re \dot{Y}(s)^* & Re Y(s)^* \end{pmatrix} = \mathcal{A}^*.$
3. $\mathcal{A}_L(s) := \begin{pmatrix} L^{\frac{1}{2}} Im \dot{Y}(s) & L^{-\frac{1}{2}} Re \dot{Y}(s) \\ L^{-\frac{1}{2}} Im Y(s) & L^{-\frac{1}{2}} Re Y(s) \end{pmatrix}$

| Intertwining operator A | $\tau_L A \tau_L^*$ |
|-------------------------------------------------------------------------------------------------------|------------------------------------------------------------------|
| $\mu(a) : \text{Adapted Model} \rightarrow \text{Twisted Model}$ | $\tau_L \mu(a) \tau_L^* = \mu(r_\alpha) \mu(a)$ |
| $\mu(r_\alpha) : \text{Model} \rightarrow \text{Twisted Model}$ | $\tau_L \mu(r_\alpha) \tau_L^* = \mu(r_\alpha(L)) \mu(r_\alpha)$ |
| With $\tilde{a}_s := a_s r_\alpha(s), \mu(\tilde{a}) : \text{Adapted Model} \rightarrow \text{Model}$ | $\tau_L \mu(\tilde{a}) \tau_L^* = \mu(\tilde{a})$ |

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